

COMBINATORIAL FRAMEWORKS FOR CLUSTER ALGEBRAS

NATHAN READING AND DAVID E SPEYER

ABSTRACT. We develop a general approach to finding combinatorial models for cluster algebras. The approach is to construct a labeled graph called a **framework**. When a framework is constructed with certain properties, the result is a model incorporating information about exchange matrices, principal coefficients, \mathbf{g} -vectors, \mathbf{g} -vector fans, and (conjecturally) denominator vectors. The idea behind frameworks arises from Cambrian combinatorics and sortable elements, and in this paper, we use sortable elements to construct a framework for any cluster algebra with an acyclic initial exchange matrix. This Cambrian framework yields a model of the entire (principal coefficients) exchange graph when the cluster algebra is of finite type. Outside of finite type, the Cambrian framework models only part of the exchange graph. In a forthcoming paper, we extend the Cambrian construction to produce a complete framework for a cluster algebra whose associated Cartan matrix is of affine type.

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1. INTRODUCTION

Cluster algebras were introduced in [10] as a tool for studying total positivity and canonical bases in semisimple algebraic groups. They have since appeared in various fields, including Teichmüller theory, Poisson geometry, quiver representations, Lie theory, algebraic geometry and algebraic combinatorics. Among the key open problems surrounding cluster algebras are various structural conjectures found in [12, 13]. (See Section 3.3 for statements of some of these conjectures.)

This paper continues a series of papers aimed at creating combinatorial models for cluster algebras within the context of Coxeter groups and root systems, using in particular the machinery of sortable elements and Cambrian (semi)lattices. Cambrian lattices were introduced in [21] as certain lattice quotients (or alternately sublattices) of the weak order on a finite Coxeter group. Conjectures in that paper, later proved in [15, 22, 23, 24], established the relevance of Cambrian lattices to Coxeter-Catalan combinatorics. In particular, Cambrian lattices are closely related to generalized associahedra, which serve as combinatorial models for cluster algebras of finite type [9, 11]. (See e.g. [1, 8] for an introduction to Coxeter-Catalan combinatorics.) In the process of proving the conjectures, sortable elements were defined in [22] and shown in [23] to provide a direct combinatorial realization of Cambrian lattices. Finally, in [25], the combinatorial theory surrounding sortable elements was further developed and extended to arbitrary Coxeter groups (from the special case of finite Coxeter groups).

In this paper, we construct Cambrian combinatorial models for cluster algebras. The insights gained from our first direct constructions of Cambrian models led to a general blueprint for building combinatorial/algebraic models. Accordingly, we begin the paper by defining the notion of a **framework** for an exchange matrix B . The matrix B defines a Cartan companion A and thus a root system. The matrix B also defines a bilinear form ω on the root space. In essence, a framework is a graph G with two labelings: each pair (v, e) consisting of a vertex contained in an edge is given a **label** $C(v, e)$ and a **co-label** $C^\vee(v, e)$. These labels are vectors in the root space that are related by a positive scaling. They should be thought of as a root and its corresponding co-root, and indeed, in important special cases, this is the case. The labeling/co-labeling must satisfy certain conditions; when it does, a detailed combinatorial model of the cluster algebra can be extracted from the framework.

The following theorem summarizes how to recover combinatorial properties of a cluster algebra from a framework for it. The precise details are Theorems 3.23 and 3.24.

Theorem 1.1. *Let B be a skew-symmetrizable acyclic integer matrix. Let \mathcal{A} be cluster algebra whose initial seed t_0 has exchange matrix B and principal coefficients. Let (G, C, C^\vee) be a framework for B . There is a map from vertices v of G to seeds t such that*

- *There is a base vertex v_b mapping to t_0 .*
- *Edges in G correspond to mutations of seeds.*
- *The exchange matrix of t has entries $[\omega(C^\vee(v, e), C(v, f))]_{e, f \ni v}$.*
- *The columns of the bottom half of the extended exchange matrix, at t , are the simple root coordinates of the vectors $C(v, e)$.*

- The \mathbf{g} -vectors of cluster variables in t , with respect to the seed t_0 , are the fundamental-weight coordinates of the basis of the weight space that is dual to the basis $\{C^\vee(v, e)\}_{e \ni v}$

Conjecturally, the framework also contains information about denominator vectors, based on a new general conjecture relating denominator vectors to \mathbf{g} -vectors: For acyclic B , we define an explicit linear map with an easily computed inverse and conjecture (as Conjecture 3.19) that the map relates denominator vectors to \mathbf{g} -vectors for cluster variables outside the initial seed. The conjecture is easily verified when B is 2×2 , and we prove the conjecture when B is of finite Cartan type (i.e. when the Cartan companion of B is of finite type).

A framework is **complete** if G is a regular graph of the correct degree, and a complete framework models the entire exchange graph of the cluster algebra. We define some other conditions on frameworks in Section 4, including the notions of an **exact** framework and a **well-connected polyhedral** framework. When the framework is complete and exact, the graph G is isomorphic to the exchange graph. When the framework is also polyhedral and well-connected, it defines a fan that coincides with the fan of \mathbf{g} -vectors. The existence of a complete, exact and/or well-connected polyhedral framework for B implies several of the conjectures from [12, 13]. See Theorem 4.1 and Corollaries 4.4 and 4.6 for details.

Frameworks are not only a convenient way to create models of cluster algebras, but in fact they are, conjecturally, the *only* way to create models. This idea is made precise in Theorem 3.27 and Corollary 3.31, which state that, assuming some conjectures from [13], every cluster algebra defines a framework. The theorem and corollary, together with the recipes in Theorem 1.1 for reading off combinatorial information from a framework imply that every model for principal coefficients and/or \mathbf{g} -vectors and/or (conjecturally) denominator vectors *is* a framework.

Comparing the method of determining \mathbf{g} -vectors from a framework to the method of determining principal coefficients, we arrive (Corollary 3.28) at an insight that also recently appeared as the first assertion of [19, Theorem 1.2]: Assuming a certain conjecture from [13], the matrix formed by \mathbf{g} -vectors is the inverse of the matrix given by principal coefficients.

The main thrust of the project, however, is to give direct proofs of the structural conjectures by constructing explicit frameworks based on sortable elements and Cambrian lattices, without relying on results from other approaches to cluster algebras. In a sense we can do this quite generally: For every acyclic B , we construct a **Cambrian framework** and prove that the framework is exact, polyhedral, and well-connected. The underlying graph is the Cambrian (semi)lattice, so that the vertices of the graph are the sortable elements. The labels are certain roots that can be read off combinatorially from the sortable elements, and the co-labels are the associated co-roots. In the Cambrian framework, the \mathbf{g} -vectors can also be read off combinatorially without having to compute a dual basis. (See Theorem 5.32.) Conjecturally, the denominator vectors can also be read off combinatorially, and this conjecture is already proven in the case where B is of finite Cartan type.

When B is of finite Cartan type, the Cambrian framework is complete, and thus defines a combinatorial and polyhedral model for the entire exchange graph. Furthermore, the existence and properties of the Cambrian framework for B of finite Cartan type imply [13, Conjecture 4.7] as well as Conjecture 3.19, described above.

The Cambrian framework also provides new proofs of many of the conjectures from [12, 13] for B of finite Cartan type. See Theorem 5.13 for details.

When B is of infinite Cartan type, the Cambrian framework is not complete. The incompleteness of the Cambrian framework is the result of a fundamental obstacle: every cone in the Cambrian fan intersects the Tits cone of the Coxeter group associated to B . But cones in the (conjectural) fan defined by \mathbf{g} -vectors do not all intersect the Tits cone. In a forthcoming paper, the authors construct a complete framework for B of affine Cartan type and use it to prove many many of the conjectures from [12, 13] for such B . In another forthcoming paper, the second author and Hugh Thomas construct a complete framework for an arbitrary skew-symmetric exchange matrix B . Their framework is constructed from the representation theory of quivers.

We conclude this introduction with two remarks about the key features of frameworks. As indicated above, the labels on a framework are essentially the columns of the bottom halves of principal-coefficients extended exchange matrices. Thus, passing from vertex to vertex, the labels need to change in a way that amounts to matrix mutation. The mutation relation is given in [13, Equation (5.9)], assuming one of the conjectures from [13]. The relation depends, of course, on the top halves of the extended exchange matrices, i.e. the exchange matrices. What is missing from [13, Equation (5.9)] is the insight that the exchange matrices themselves are determined from the labels, as described in Theorem 1.1. Thus [13, Equation (5.9)] is replaced by a self-contained mutation rule in terms only of the labels. This rule is called the **Transition condition**, and is the most important feature of the notion of a framework. In important cases, all of the labels are real roots, the co-labels are the corresponding co-roots, and the Transition condition can be replaced by the **Reflection condition**. This condition says when u and v are connected by an edge and t is the reflection associated to $C(v, e)$, each label on u is either identical to a label on v , or is obtained from a label on v by the action of t . The Reflection condition was discovered in connection with combinatorial/polyhedral investigations of Cambrian fans.

Another technical feature of frameworks that should not be overlooked is the use of root and co-root lattices (and thus weight and co-weight lattices) to handle the possible absence of skew-symmetry in B . Simply by placing vectors in the correct lattice (i.e. deciding whether to write the prefix “co”), the difficulties caused by asymmetric B completely disappear from the general theory of frameworks. This is exactly analogous to the purpose of roots and co-roots, etc. in handling asymmetric Cartan matrices. Given the fact that a Cambrian model exists for every acyclic B , based on the Cartan companion A , this analogy is hardly accidental.

2. FRAMEWORKS AND REFLECTION FRAMEWORKS

In this section, we define the general notion of a framework and a special kind of framework called a reflection framework.

2.1. Frameworks. The exchange matrix $B = [b_{ij}]$ associated to a cluster algebra is a skew-symmetrizable integer matrix, with rows and columns indexed by a set I , with $|I| = n$. That means that there exists a positive real-valued function δ on I such that $\delta(i)b_{ij} = -\delta(j)b_{ji}$ for all $i, j \in I$. Let A be the matrix with diagonal entries 2 and off-diagonal entries $a_{ij} = -|b_{ij}|$. Then A is a symmetrizable generalized

Cartan matrix in the sense of [16] (see also [25, Section 2.2]), called the **Cartan companion** of B . In particular, $\delta(i)a_{ij} = \delta(j)a_{ji}$ for all $i, j \in I$.

Let V be a real vector space of dimension n with a basis $\Pi = \{\alpha_i : i \in I\}$ and let V^* be the dual vector space. The set Π is called the set of **simple roots**. The canonical pairing between $x \in V^*$ and $y \in V$ is written $\langle x, y \rangle$. We set $\alpha_i^\vee = \delta(i)^{-1}\alpha_i$. The vectors α_i^\vee are called the **simple co-roots** and the set of simple co-roots is written Π^\vee . We write D for the **fundamental domain**, $\bigcap_{i \in I} \{x \in V^* : \langle x, \alpha_i \rangle \geq 0\}$.

The exchange matrix B defines a bilinear form ω by setting $\omega(\alpha_i^\vee, \alpha_j) = b_{ij}$. The form ω is skew-symmetric, because

$$\omega(\alpha_i^\vee, \alpha_j) = b_{ij} = -\frac{\delta(j)}{\delta(i)}b_{ji} = -\frac{\delta(j)}{\delta(i)}\omega(\alpha_j^\vee, \alpha_i) = -\omega(\alpha_j, \alpha_i^\vee).$$

A **quasi-graph** is a hypergraph with edges of size 1 or 2. Each edge of size two is an edge in the usual graph-theoretical sense, while an edge of size 1 should be thought of as dangling from a vertex, and thus not connecting that vertex to any other. We will sometimes refer to edges of size 1 as **half-edges** and to edges of size 2 as **full edges**. Half-edges should not be confused with “self-edges” or “loops,” i.e. edges that connect a vertex to itself. In this paper, all quasi-graphs will be **simple**, meaning that no two edges connect the same pair of vertices, and that every edge of size two connects two distinct vertices. The **degree** of a vertex in a simple quasi-graph is the total number of edges (including half-edges) incident to that vertex, and the quasi-graph is **regular of degree n** if every vertex has degree n . A quasi-graph G is **connected** if the graph obtained from G by ignoring half-edges is connected in the usual sense.

An **incident pair** in a quasi-graph G is a pair (v, e) where v is a vertex contained in an edge e . For each vertex v , let $I(v)$ denote the set of edges e such that (v, e) is an incident pair. A **framework for B** will be a triple (G, C, C^\vee) , where G is a connected quasi-graph that is regular of degree n (where B is $n \times n$) and each of C and C^\vee is a labeling of each incident pair in G by a vector in V , satisfying certain conditions given below. The **label** on (v, e) will be written $C(v, e)$, and the notation $C(v)$ will stand for the set $\{C(v, e) : e \in I(v)\}$. The **co-label** on (v, e) will be written $C^\vee(v, e)$, and the notation $C^\vee(v)$ will stand for the set $\{C^\vee(v, e) : e \in I(v)\}$.

Co-label condition: For each incident pair (v, e) , the co-label $C^\vee(v, e)$ is a positive scalar multiple of the label $C(v, e)$.

We will see below that in an important special case, each label $C(v, e)$ will be a real root, and each co-label $C^\vee(v, e)$ will be the associated co-root $(C(v, e))^\vee$. In general the label $C(v, e)$ need not be a real root, so there is no available notion of a co-root associated to $C(v, e)$. Despite the fact that $C^\vee(v, e)$ may not be a co-root in any meaningful sense, when $\beta = C(v, e)$, we will write β^\vee for $C^\vee(v, e)$.

Sign condition: For each incident pair (v, e) :

- (1) The label $C(v, e)$ is not the zero vector; and
- (2) Either $C(v, e)$ or $-C(v, e)$ is in the nonnegative span of the simple roots.

Assuming the Sign condition, each label has a well-defined **sign** $\text{sgn}(C(v, e)) \in \{\pm 1\}$, namely $\text{sgn}(C(v, e)) = 1$ if $C(v, e)$ is in the nonnegative span of the simple

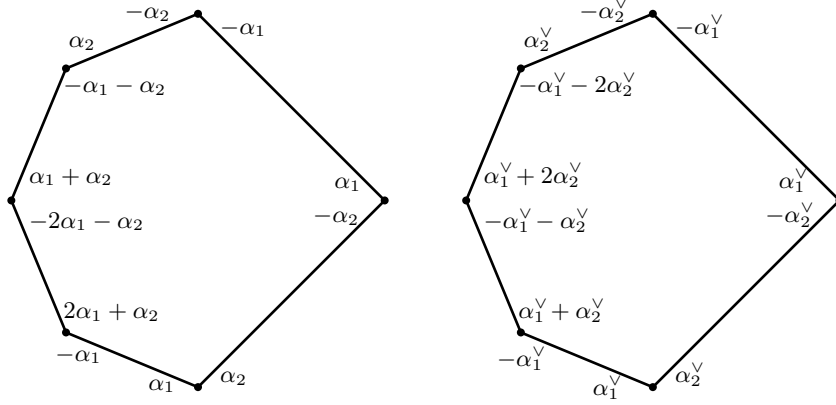


FIGURE 1. A framework for the exchange matrix $B = \begin{pmatrix} 0 & -2 \\ -1 & 0 \end{pmatrix}$

roots, or $\text{sgn}(C(v, e)) = -1$ if $-C(v, e)$ is in the nonnegative span of the simple roots. We also define $\text{sgn}(C^\vee(v, e)) = \text{sgn}(C(v, e))$.

Base condition: There exists a vertex v_b such that $C(v_b)$ is the set of simple roots and $C^\vee(v_b)$ is the set of simple co-roots.

The Base condition lets us identify the indexing set I with $I(v_b)$ by identifying $e \in I(v_b)$ with the index $i \in I$ such that $C(v_b, e) = \alpha_i$.

We use the notations $[x]_+ = \max(x, 0)$ and $[x]_- = \min(x, 0)$.

Transition condition: Suppose v and v' are distinct vertices incident to the same edge e . Then $C(v, e) = -C(v', e)$. Furthermore, if $\beta = C(v, e)$ and $\gamma \in C(v) \setminus \{\beta\}$, then $\gamma + [\text{sgn}(\beta)\omega(\beta^\vee, \gamma)]_+ \beta$ is in $C(v')$.

Co-transition condition: Suppose v and v' are distinct vertices incident to the same edge e . Then $C^\vee(v, e) = -C^\vee(v', e)$. Furthermore, if $\beta^\vee = C^\vee(v, e)$ and $\gamma^\vee \in C^\vee(v) \setminus \{\beta^\vee\}$, then $\gamma^\vee + [-\text{sgn}(\beta^\vee)\omega(\gamma^\vee, \beta)]_+ \beta^\vee$ is in $C^\vee(v')$.

The pair (G, C, C^\vee) will be called a **framework for B** if G is connected and if the triple satisfies the Co-label, Sign, Base, Transition, and Co-transition conditions.

Example 2.1. We now construct a framework for the non-skew-symmetric exchange matrix $B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$. This is skew-symmetrizable with $\delta(2) = 2\delta(1)$. The Cartan companion of B is $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$. The skew-symmetric form ω is given by $\omega(\alpha_1^\vee, \alpha_2) = 2$ and $\omega(\alpha_2^\vee, \alpha_1) = -1$. A framework for B is shown in Figure 1. The figure shows two copies of the same graph, with labels on one and co-labels on the other. The label on an incident pair (v, e) is shown near e , closer to v than to the other vertex of e . The vertex at the bottom is v_b .

Remark 2.2. When B is skew-symmetric, rather than merely skew-symmetrizable, each simple co-root equals the corresponding simple root. By a simple inductive argument, a framework (G, C, C^\vee) for B must have $C^\vee(v, e) = C(v, e)$ for every incident pair (v, e) . Thus, for skew-symmetric B , we may as well define a framework

to be a pair (G, C) satisfying the Sign condition, the Base condition (ignoring the requirement about C^\vee), and the Transition condition (replacing β^\vee by β).

We now establish some first properties of frameworks.

Proposition 2.3. *Suppose (G, C, C^\vee) is a framework for B and let v be a vertex of G . Then the label set $C(v)$ is a basis for the root lattice and $C^\vee(v)$ is a basis for the co-root lattice.*

Proof. By the Base condition, the proposition holds for $v = v_b$. If $v \neq v_b$, then since G is connected, there is a finite path from v to v_b . The Transition condition says that at each step in the path, the set $C(\cdot)$ changes by a sequence of Gauss-Jordan operations that alter a label by adding an integer multiple of another label. Thus each step preserves the property of being a basis for the root lattice. The assertion for $C(v)$ follows by an easy induction, and the assertion for $C^\vee(v)$ follows by the analogous proof. \square

Proposition 2.3 also allows us to define **mutations** of edges. The use of the term “mutation” in this context is inspired by, and will be compatible with, the use of the term in connection with cluster algebras. Let e be a full edge connecting v to v' . We will define a function μ_e from $I(v)$ to $I(v')$. Define $\mu_e(e)$ to be e . If $f \in I(v) \setminus \{e\}$, then define $\mu_e(f)$ to be the edge $f' \in I(v')$ such that $C(v', f') = \gamma + [\text{sgn}(\beta)\omega(\beta^\vee, \gamma)]_+ \beta$. The notion of mutations of edge sets allows us to make the Transition condition slightly more specific, by identifying roots in $C(v)$ with roots in $C(v')$ in terms of edge mutations.

Transition condition, strengthened: Suppose v and v' are distinct vertices incident to the same edge e . Then $C(v, e) = -C(v', e)$. Furthermore, if $f \in I(v) \setminus \{e\}$, then

$$C(v', \mu_e(f)) = C(v, f) + [\text{sgn}(C(v, e))\omega(C^\vee(v, e), C(v, f))]_+ C(v, e).$$

The following proposition is immediate by the definition of μ_e .

Proposition 2.4. *A framework for B satisfies the strengthened Transition condition.*

We strengthen the Co-transition condition similarly, as follows:

Co-transition condition, strengthened: Suppose v and v' are distinct vertices incident to the same edge e . Then $C^\vee(v, e) = -C^\vee(v', e)$. Furthermore, if $f \in I(v) \setminus \{e\}$, then

$$C^\vee(v', \mu_e(f)) = C^\vee(v, f) + [-\text{sgn}(C^\vee(v, e))\omega(C^\vee(v, f), C(v, e))]_+ C^\vee(v, e).$$

Proposition 2.5. *A framework for B satisfies the strengthened Co-transition condition.*

Proposition 2.5 is not immediate like Proposition 2.4, because, *a priori*, we do not know that an edge mutation operation defined in terms of the Co-transition condition would agree with μ_e . The Co-label condition allows us to avoid this difficulty.

Proof. Suppose v and v' are distinct vertices incident to the same edge e . The assertion that $C^\vee(v, e) = -C^\vee(v', e)$ is part of the (unstrengthened) Co-transition

condition. Furthermore, if $f \in I(v) \setminus \{e\}$, then the Co-label condition and the strengthened Transition condition imply that $C^\vee(v', \mu_e(f))$ is a positive scalar multiple of

$$(2.1) \quad C(v, f) + [\operatorname{sgn}(C(v, e))\omega(C^\vee(v, e), C(v, f))]_+ C(v, e).$$

The (unstrengthened) co-Transition condition says that $C^\vee(v')$ contains the vector

$$(2.2) \quad C^\vee(v, f) + [-\operatorname{sgn}(C^\vee(v, e))\omega(C^\vee(v, f), C(v, e))]_+ C^\vee(v, e).$$

Writing $C^\vee(v, e) = aC(v, e)$ and $C^\vee(v, f) = bC(v, f)$, we rewrite (2.1) as

$$(2.3) \quad C(v, f) + [\operatorname{sgn}(C(v, e))\omega(aC(v, e), C(v, f))]_+ C(v, e).$$

and rewrite (2.2) as

$$(2.4) \quad bC(v, f) + [-\operatorname{sgn}(C(v, e))\omega(bC(v, f), C(v, e))]_+ aC(v, e).$$

Using the antisymmetry and linearity of ω , we see that (2.4) is b times (2.1). Proposition 2.3 implies that only one multiple of (2.1) is in $C^\vee(v')$, so we conclude that (2.2) is $C^\vee(v', \mu_e(f))$. \square

In light of Propositions 2.4 and 2.5, we will tacitly use the strengthened forms of the Transition and Co-transition conditions when needed. The point is that the original statements of the conditions are easier to state and potentially easier to check, but that the strengthened statements provide more precise control for use in arguments.

Remark 2.6. The proof of Proposition 2.5 also establishes the following fact: The ratio between $C^\vee(v', \mu_e(f))$ and $C(v', \mu_e(f))$ equals the ratio between $C^\vee(v, f)$ and $C(v, f)$. Thus the co-label-to-label ratios appearing throughout a framework are exactly the ratios between simple co-roots and simple roots.

The dual bases to the co-label sets $C^\vee(v)$ will be of great importance. Given a framework (G, C, C^\vee) , and a vertex v of G , let $R(v)$ be the basis of V^* that is dual to the basis $C^\vee(v)$ of V . More specifically, for each $e \in I(v)$, let $R(v, e)$ be the basis vector in $R(v)$ that is dual to $C^\vee(v, e)$.

Proposition 2.7. *Let (G, C, C^\vee) be a framework for B and let v and v' be adjacent vertices of G . Then $R(v) \cap R(v')$ contains exactly $n - 1$ vectors. Specifically, if e is the edge connecting v to v' and $f \in I(v) \setminus \{e\}$, then $R(v, f) = R(v', \mu_e(f))$. Also, $R(v, e)$ and $R(v', e)$ lie on opposite sides of the hyperplane spanned by $R(v) \cap R(v')$.*

Proof. Let $f \in I(v) \setminus \{e\}$. We know $\langle R(v, f), C^\vee(v, p) \rangle = \delta_{f,p}$ (Kronecker delta) for $p \in I(v)$. The Co-transition condition says, in particular, that $C^\vee(v, p)$ and $C^\vee(v', \mu_e(p))$ differ by multiples of $C^\vee(v, e)$. Thus $\langle R(v, f), C^\vee(v', \mu_e(p)) \rangle$ equals $\langle R(v, f), C^\vee(v, p) \rangle = \delta_{f,p}$ for $p \in I(v)$. Now $R(v, f) = R(v', \mu_e(f))$, because $R(v, f)$ satisfies the conditions that define $R(v', \mu_e(f))$.

Finally, by the definition of a dual basis, the $(n-1)$ -plane spanned by $R(v) \cap R(v')$ is $C^\vee(v, e)^\perp$. Since, by the definition of a dual basis, $\langle C^\vee(v, e), R(v, e) \rangle = 1$ and $\langle C^\vee(v, e), R(v', e) \rangle = \langle -C^\vee(v', e), R(v, e) \rangle = -1$, we see that $R(v, e)$ and $R(v', e)$ are on opposite sides of this plane. \square

We define $\operatorname{Cone}(v)$ to be the simplicial cone in V^* spanned by the $R(v, e)$, as e ranges over the neighbors of v , so $\operatorname{Cone}(v) = \bigcap_{e \in I(v)} \{x \in V^* : \langle x, C^\vee(v, e) \rangle \geq 0\}$. Proposition 2.7 has the following corollary.

Corollary 2.8. *The cones $\text{Cone}(v)$ and $\text{Cone}(v')$ intersect in a common facet.*

We conclude with the following observation:

Proposition 2.9. *If (G, C, C^\vee) is a framework for B then (G, C^\vee, C) is a framework for $-B^T$.*

In the notation $-B^T$, the superscript T denotes transpose. Thus moving from B to $-B^T$ should be thought of as “transposing the absolute values but not the signs.” The exchange matrix B has Cartan companion A , which defines simple roots Π and simple co-roots Π^\vee . The exchange matrix $-B^T$ has Cartan companion A^T , which defines simple roots Π^\vee and simple co-roots Π . Proposition 2.9 is immediate from the definition, once we switch the roles of simple roots and simple co-roots, and switch the roles of the Transition condition and Co-transition condition.

2.2. Reflection frameworks. We now define a special kind of framework called a reflection framework, in which all labels are roots and co-roots, and the Transition condition can be rephrased in terms of the action of reflections in the Coxeter group.

The **Euler form** E associated to B is defined on the bases of simple roots and co-roots as follows:

$$E(\alpha_i^\vee, \alpha_j) = \begin{cases} [b_{ij}]_- & \text{if } i \neq j, \text{ or} \\ 1 & \text{if } i = j. \end{cases}$$

Recall that the Cartan companion A of B is the matrix with diagonal entries 2 and off-diagonal entries $a_{ij} = -|b_{ij}|$, where $B = [B_{ij}]_{i,j \in I}$. Define a bilinear form K on V by $K(\alpha_i^\vee, \alpha_j) = a_{ij}$. The form K is symmetric. (The proof is essentially identical to the proof that ω is anti-symmetric.)

Proposition 2.10. *The form ω is given by $\omega(\beta, \gamma) = E(\beta, \gamma) - E(\gamma, \beta)$, and the form K is given by $K(\beta, \gamma) = E(\beta, \gamma) + E(\gamma, \beta)$.*

Proof. We will check these identities for $\beta = \alpha_i^\vee$ and $\gamma = \alpha_j$. If $i = j$, then the identities are easy, so suppose $i \neq j$. Then $E(\alpha_i^\vee, \alpha_j) - E(\alpha_j, \alpha_i^\vee)$ equals

$$E(\alpha_i^\vee, \alpha_j) - \frac{\delta(j)}{\delta(i)} E(\alpha_j^\vee, \alpha_i) = \min(b_{ij}, 0) - \frac{\delta(j)}{\delta(i)} \min(b_{ji}, 0),$$

which equals $\min(b_{ij}, 0) + \max(b_{ij}, 0) = b_{ij}$ because $\delta(i)b_{ij} = -\delta(j)b_{ji}$. This is the first identity.

Similarly, $E(\alpha_i^\vee, \alpha_j) + E(\alpha_j, \alpha_i^\vee) = \min(b_{ij}, 0) - \max(b_{ij}, 0) = -|b_{ij}| = a_{ij}$. \square

The following is an immediate corollary of Proposition 2.10.

Corollary 2.11. *If $\beta, \gamma \in V$ then $|\omega(\beta, \gamma)| = |K(\beta, \gamma)|$ if and only if either $E(\beta, \gamma) = 0$ or $E(\gamma, \beta) = 0$ or both.*

The Cartan matrix A defines a Coxeter group W generated by $S = \{s_i : i \in I\}$ whose defining relations $(s_i s_j)^{m(i,j)}$ are given by

$$m(s, s') = \begin{cases} 2 & \text{if } a_{ij} \cdot a_{ji} = 0, \\ 3 & \text{if } a_{ij} \cdot a_{ji} = 1, \\ 4 & \text{if } a_{ij} \cdot a_{ji} = 2, \\ 6 & \text{if } a_{ij} \cdot a_{ji} = 3, \text{ or} \\ \infty & \text{if } a_{ij} \cdot a_{ji} \geq 4. \end{cases}$$

The Cartan matrix also defines an action of W on V . The action of a generator $s_i \in S$ on a simple root α_j is $s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$ and the action on a simple co-root α_j^\vee is $s_i(\alpha_j^\vee) = \alpha_j^\vee - a_{ji}\alpha_i^\vee$. The action of W preserves the form K .

A **real root** β is a vector in the orbit, under the action of the Coxeter group W , of some simple root. **Real co-roots** are defined similarly. The **real root system** Φ associated to A is the set of all real roots. A root is **positive** if it is in the nonnegative linear span of the simple roots. Otherwise, it is in the nonpositive linear span of the simple roots and is called **negative**. The **reflections** in W are the elements conjugate to elements of S . There is a bijection $t \rightarrow \beta_t$ from reflections to positive roots and a bijection $t \rightarrow \beta_t^\vee$ from reflections to positive co-roots such that t acts on V by sending $x \in V$ to $tx = x - K(\beta_t^\vee, x)\beta_t$. The action of W on V commutes with passing from roots to co-roots: That is, $w(\beta^\vee) = (w\beta)^\vee$ for any $w \in W$ and any root β .

The set of **almost positive roots** is the union of the set of positive roots and the set of negative simple roots. Similarly, the almost positive co-roots are those co-roots that are either positive or negative simple. If V is two dimensional, then the almost positive roots are all vectors in a two dimensional vector space, no two of which are positive multiples of each other, so we may consider them to be cyclically ordered as they wind around the origin. This concept and ordering will return later in the paper.

Let G be a connected quasi-graph and let C be a labeling of each incident pair in G by a vector in V . We define some additional conditions on (G, C) .

Root condition: Each label $C(v, e)$ is a real root with respect to the Cartan matrix A .

A pair (G, C) satisfying the Root condition automatically satisfies the Sign condition. Suppose v is a vertex of G . Define $C_+(v)$ to be the set of positive roots in $C(v)$ and define $C_-(v)$ to be the set of negative roots in $C(v)$. Let $\Gamma(v)$ be the directed graph whose vertex set is $C(v)$, with an edge $\beta \rightarrow \beta'$ if $E(\beta, \beta') \neq 0$.

Euler conditions: Suppose v is a vertex of G and let e and f be distinct edges incident to v . Write $\beta = C(v, e)$ and $\gamma = C(v, f)$. Then

- (E0) At least one of $E(\beta, \gamma)$ and $E(\gamma, \beta)$ is zero.
- (E1) If $\beta \in C_+(v)$ and $\gamma \in C_-(v)$ then $E(\beta, \gamma) = 0$.
- (E2) If $\text{sgn}(\beta) = \text{sgn}(\gamma)$ then $E(\beta, \gamma) \leq 0$.
- (E3) The graph $\Gamma(v)$ is acyclic.

Reflection condition: Suppose v and v' are distinct vertices incident to the same edge e . If $\beta = C(v, e) = \pm\beta_t$ for some reflection t and $\gamma \in C(v)$, then $C(v')$ contains the root

$$\gamma' = \begin{cases} t\gamma & \text{if } \omega(\beta_t^\vee, \gamma) \geq 0, \text{ or} \\ \gamma & \text{if } \omega(\beta_t^\vee, \gamma) < 0. \end{cases}$$

Applying the Reflection condition with $\gamma = \beta$, we see that $-\beta \in C(v')$. Condition (E0) implies, in light of Proposition 2.10, that $\omega(\beta_t^\vee, \gamma)$ and $K(\beta_t^\vee, \gamma)$ agree in absolute value. Thus except in the case $\gamma = \beta$, we can replace the “<” sign by “ \leq ” in the Reflection condition. Since γ and $t\gamma$ differ by a multiple of β , and since

ω is antisymmetric, the Reflection condition is symmetric in v and v' . In particular, for any edge, it is enough to check the condition in one direction. Just as we strengthened the Transition condition, we can strengthen the Reflection condition, by defining μ_e so that the strengthened condition holds.

The pair (G, C) is a **reflection framework** if it satisfies the Base condition (ignoring the assertion about the labeling C^\vee), the Root condition, the Reflection conditions and the Euler conditions (E1), (E2) and (E3). Condition (E0) follows immediately from condition (E3). The pair (G, C) is a **weak reflection framework** if it satisfies the Base condition, the Root condition, the Reflection conditions and conditions (E0), (E1) and (E2).

If a reflection framework exists for B , then in particular B is acyclic in the usual sense for exchange matrices: Namely that the directed graph on I , with directed edges $i \rightarrow j$ if and only if $b_{ij} < 0$, is acyclic. The acyclicity follows from (E3) because the directed graph on I is isomorphic to $\Gamma(v_b)$ by the Base condition. By contrast, weak reflection frameworks can exist when B is not acyclic.

The following proposition relates reflection frameworks to frameworks.

Proposition 2.12. *Suppose (G, C) is a weak reflection framework and let C^\vee be the labeling of incident pairs of G such that $C^\vee(v, e)$ is the co-root associated to $C(v, e)$. Then (G, C, C^\vee) is a framework.*

Proof. The Co-label condition holds by the definition of C^\vee , the Sign condition holds by the Root condition, and the Base condition holds by hypothesis and by the definition of C^\vee . We need to verify the Transition and Co-Transition conditions.

Suppose v is a vertex connected, by an edge e , to another vertex v' . By the Root condition, $C(v, e)$ is a root $\beta = \pm\beta_t$. By the anti-symmetry of ω we see that $\omega(\beta_t, \beta) = 0$, so the Reflection condition says, in particular, that $C(v', e) = t\beta = -\beta$, so the first condition of the Transition condition holds. Furthermore, take f to be an edge, distinct from e , in $I(v)$. Write γ for $C(v, f)$ and γ' for $C(v', \mu_e(f))$. By the (strengthened) Reflection condition, $\gamma' = t\gamma = \gamma - K(\beta^\vee, \gamma)\beta$ if $\omega(\beta_t, \gamma) \geq 0$ or $\gamma' = \gamma$ if $\omega(\beta_t, \gamma) < 0$. The (strengthened) Transition condition is the assertion that $\gamma' = \gamma + [\omega(\beta_t, \gamma)]_+ \beta$, so we have established this condition when $\omega(\beta_t, \gamma) < 0$, and it remains to establish the following: If $\omega(\beta_t, \gamma) \geq 0$ then $\omega(\beta_t, \gamma) = -K(\beta, \gamma)$. Suppose $\omega(\beta_t, \gamma) \geq 0$ and consider the four cases given by the signs of β and γ .

If β is positive, then $\beta = \beta_t$, so $E(\beta, \gamma) - E(\gamma, \beta) = \omega(\beta_t, \gamma) \geq 0$. If γ is also positive, then Condition (E2) says that $E(\beta, \gamma) \leq 0$ and $E(\gamma, \beta) \leq 0$, and by condition (E0) we conclude that $E(\beta, \gamma) = 0$. If γ is negative, then condition (E1) says that $E(\beta, \gamma) = 0$. In either case, $\omega(\beta_t, \gamma) = \omega(\beta, \gamma) = -E(\gamma, \beta)$ and $K(\beta, \gamma) = E(\gamma, \beta)$.

If β is negative, then $\beta = -\beta_t$, so $E(\beta, \gamma) - E(\gamma, \beta) = -\omega(\beta_t, \gamma) \leq 0$. If γ is also negative, then again, $E(\beta, \gamma) \leq 0$ and $E(\gamma, \beta) \leq 0$, so this time $E(\gamma, \beta) = 0$. If γ is positive, then $E(\gamma, \beta) = 0$ by condition (E1), so $\omega(\beta_t, \gamma) = -\omega(\beta, \gamma) = -E(\beta, \gamma)$ and $K(\beta, \gamma) = E(\beta, \gamma)$.

The reflection condition implies that $(\gamma')^\vee = t(\gamma^\vee) = \gamma^\vee - K(\beta^\vee, \gamma^\vee)\beta = \gamma^\vee - K(\gamma^\vee, \beta)\beta^\vee$. The Co-transition condition now follows by a similar argument. \square

In fact, the proof above of Proposition 2.12 establishes that, when the Root condition and the Euler conditions hold, the Transition condition, the Reflection condition, and the Co-transition conditions are all equivalent. The proof still goes

through under a slight weakening of the Euler conditions: We only need conditions (E0)–(E2) in the case where at least one of the edges e and f is a full edge.

Example 2.13. The framework described in Example 2.1 is a reflection framework.

3. CLUSTER ALGEBRAS AND FRAMEWORKS

In this section, we review background material on cluster algebras, show how frameworks are combinatorial models for cluster algebras, and establish the properties of these models.

3.1. Cluster algebras. As before, let I be a finite indexing set with $|I| = n$ and let B be a skew-symmetrizable integer matrix, with rows and columns indexed by I . Let \mathbb{P} be a semifield (an abelian multiplicative group with a second commutative, associative operation \oplus such that the group multiplication distributes over \oplus). Let $Y = (y_i : i \in I)$ be a tuple of elements of \mathbb{P} . Let \mathcal{F} be the field of rational functions in n indeterminates with coefficients in $\mathbb{Q}\mathbb{P}$. Let $X = (x_i : i \in I)$ be algebraically independent elements of \mathcal{F} .

Let T be the n -regular tree. We will continue the graph notation from Section 2. For each pair v, v' of vertices of T , connected by the edge e , let μ_e be a bijection from the set $I(v)$ of edges incident to v to the set $I(v')$. This defines two maps called μ_e for each edge e . We will let the context distinguish the two, and we require that $\mu_e : I(v') \rightarrow I(v)$ is the inverse of $\mu_e : I(v) \rightarrow I(v')$.

Distinguish a vertex v_b of T and identify I with the set $I(v_b)$ of edges incident to v_b . The data of (B, Y, X) constitute the **initial seed** of the cluster algebra that we define below. The matrix B is the **exchange matrix** in the seed, the elements y_i are the **coefficients** in the seed and the tuple $X = (x_i : i \in I)$ is the **cluster**, with the individual elements x_i called **cluster variables**. More generally, a **seed** is any triple consisting of an exchange matrix with rows and columns indexed by I' (for $|I'| = n$), a tuple $(y'_i : i \in I)$ of coefficients and a cluster $(x'_i : i \in I')$ of algebraically independent elements of \mathcal{F} .

We define $(B^{v_b}, Y^{v_b}, X^{v_b}) = (B, Y, X)$ and further overload the notation μ_e to define **seed mutations**. These seed mutations inductively associate a seed to each vertex of T . The indexing set for the seed at v is the set $I(v)$ of edges incident to v . Let $B^v = [b_{pq}^v]_{p,q \in I(v)}$ be the exchange matrix associated to v , let $Y^v = (y_p^v : p \in I(v))$ be the coefficients, and let $X^v = (x_p^v : p \in I(v))$ be the cluster.

Matrix mutation. Let e be an edge v to v' and define $B^{v'} = \mu_e(B^v)$ by setting

$$(3.1) \quad b_{\mu_e(p)\mu_e(q)}^{v'} = \begin{cases} -b_{pq}^v & \text{if } p = e \text{ or } q = e \\ b_{pq}^v + \text{sgn}(b_{pe}^v)[b_{pe}^v b_{eq}^v]_+ & \text{otherwise} \end{cases}$$

Coefficient mutation. Let e be an edge v to v' and define $Y^{v'} = \mu_e(Y^v)$ by setting

$$(3.2) \quad y_{\mu_e(p)}^{v'} = \begin{cases} (y_e^v)^{-1} & \text{if } p = e \\ y_p^v (y_e^v)^{[b_{ep}^v]_+} (y_e^v \oplus 1)^{-b_{ep}^v} & \text{if } p \in I(v) \setminus \{e\} \end{cases}$$

Cluster mutation. Let e be an edge v to v' and define $X^{v'} = \mu_e(X^v)$ by setting

$$(3.3) \quad x_p^{v'} = \begin{cases} \frac{1}{(y_e^v \oplus 1)x_e^v} \left(y_e^v \prod_p (x_p^v)^{[b_{pe}^v]_+} + \prod_p (x_p^v)^{[-b_{pe}^v]_+} \right) & \text{if } p = e \\ x_p^v & \text{if } p \neq e. \end{cases}$$

In both products above, p ranges over the set $I(v)$.

Seed mutation. Let e be an edge v to v' and define $(B^{v'}, Y^{v'}, X^{v'})$ to be $\mu_e(B^v, Y^v, X^v) = (\mu_e(B^v), \mu_e(Y^v), \mu_e(X^v))$. Seed mutation is involutive. That is, $\mu_e(\mu_e(B^v, Y^v, X^v)) = (B^v, Y^v, X^v)$.

The **cluster algebra** $\mathcal{A}(B, Y, X)$ is the subalgebra of \mathcal{F} generated by all of the cluster variables y_p^v where v ranges over all vertices of T and p ranges over $I(v)$. The cluster algebra is determined by any of its seeds (B^v, Y^v, X^v) . Up to isomorphism, it is determined any of the pairs (B^v, Y^v) .

Two vertices $v, v' \in T$ define the same seed if there is a bijection $\lambda : I(v) \rightarrow I(v')$ such that, first $b_{\lambda(e)\lambda(f)}^{v'} = b_{ef}^v$, second $b_{\lambda(e)}^{v'} = b_e^v$, and third $x_{\lambda(e)}^{v'} = x_e^v$ for all $e, f \in I(v)$. When such an λ exists, it is unique, and furthermore, by seed mutation it induces a bijection from the neighbors of v to the neighbors of v' such that each neighbor of v defines the same seed as the corresponding neighbor of v' . Thus we can define a quotient of T by identifying two vertices v and v' if they define the same seed, and identifying edges of v with edges of v' according to the map λ . This quotient is the **exchange graph** $\text{Ex}(B, Y, X)$ associated to (B, Y, X) . The mutation maps μ_e are compatible with this quotient. Thus we can think of μ_e as a map, for each pair v, v' of vertices of the exchange graph connected by the edge e , from the set $I(v)$ of edges incident to v in the exchange graph to the set $I(v')$ of edges incident to v' in the exchange graph. In all of the notation defined above, we can safely replace the n -regular tree T by the exchange graph $\text{Ex}(B, Y, X)$.

The first surprising theorem [10, Theorem 3.1] about cluster algebras is the following result, known as the Laurent phenomenon:

Theorem 3.1. *For any vertex v in $\text{Ex}(B, Y, X)$, the cluster algebra $\mathcal{A}(B, Y, X)$ is contained in the ring $\mathbb{Q}[x_i^v : i \in I(v)]$. In other words, if x is any cluster variable in any cluster, then x can be uniquely written as*

$$(3.4) \quad x = \frac{N(x_i^v : i \in I(v))}{\prod_{i \in I(v)} (x_i^v)^{d_i}},$$

where the d_i are integers and N is a polynomial in the variables $(x_i^v : i \in I(v))$ with coefficients in $\mathbb{Z}\mathbb{P}$ which is not divisible by any of the variables x_i^v .

The vector $\mathbf{d}^v(x) = \sum_{i \in I} d_i \alpha_i$ in the root lattice is the **denominator vector** of x with respect to the vertex v . Usually, the denominator vector is defined to be the integer vector $(d_i : i \in I(v))$, which can be recovered from $\mathbf{d}^v(x)$ by taking simple root coordinates. We will only consider denominator vectors with respect to the vertex v_b , so we will use the abbreviation $\mathbf{d}(x)$ for $\mathbf{d}^{v_b}(x)$.

The most important instances of cluster algebras are the cluster algebras of geometric type. Let J be an indexing set, disjoint from I , with $|J| = m$ and let $(x_j : j \in J)$ be independent variables. Let $\mathbb{P} = \text{Trop}(x_j : j \in J)$ be the **tropical semifield** generated by $(x_j : j \in J)$. This is the free abelian group generated by $(x_j : j \in J)$, written multiplicatively, with an addition operation \oplus defined by

$$\prod_{j \in J} x_j^{a_j} \oplus \prod_{j \in J} x_j^{b_j} = \prod_{j \in J} x_j^{\min(a_j, b_j)}.$$

Let $\tilde{B} = [b_{ij}]$ be an integer matrix with rows indexed by the disjoint union $I \sqcup J$ and columns indexed by I such that B is the $n \times n$ matrix consisting of the rows

of \tilde{B} indexed by I . Such a matrix is called an **extended exchange matrix**. The rows of \tilde{B} indexed by J specify a tuple $(y_i : i \in I)$ of elements of \mathbb{P} by setting $y_i = \prod_{j \in J} x_j^{b_{ji}^v}$, for each $i \in I$. Thus a pair (\tilde{B}, X) encodes a seed. Following the construction from above, we associate a seed (\tilde{B}^v, X^v) to each vertex. The cluster algebra generated in this way is called a **cluster algebra of geometric type**. The matrix \tilde{B}^v has rows indexed by $I(v) \cup J$ and columns indexed by $I(v)$. If v and v' are connected by an edge e , the relationship between the extended exchange matrices \tilde{B}^v and $\tilde{B}^{v'}$ is given by the matrix mutation relation (3.1), where p is in $I(v) \cup J$ rather than $I(v)$. In particular, coefficient mutation does not need to be treated separately, but coefficients associated to a vertex v can still be recovered as $y_i^v = \prod_{j \in J} x_j^{b_{ji}^v}$. Mutation of clusters can also be written more simply.

Cluster mutation (geometric type). Let e be an edge v to v' and define $X^{v'} = \mu_e(X^v)$ by setting

$$(3.5) \quad x_p^{v'} = \begin{cases} \frac{1}{x_e^v} \left(\prod_p (x_p^v)^{[b_{pe}^v]_+} + \prod_p (x_p^v)^{[-b_{pe}^v]_+} \right) & \text{if } p = e \\ x_p^v & \text{if } p \neq e. \end{cases}$$

in both products, p now ranges over the set $I(v) \cup J$ rather than the set $I(v)$.

Of primary importance among cluster algebras of geometric type are the cluster algebras with **principal coefficients**. In this case we take J to be (a disjoint copy of) I , so that the initial extended exchange matrix \tilde{B} is a $2n \times n$ matrix with B in the top n rows. The bottom n rows are taken to be the $n \times n$ identity matrix. In general, we write \tilde{B}^v as $\begin{pmatrix} B^v \\ H^v \end{pmatrix}$ where B^v is the exchange matrix associated to v as before and H^v is a matrix with rows indexed by I and columns indexed by $I(v)$. Let $\mathcal{A}_0(B)$ and $\text{Ex}_0(B)$ be the cluster algebra $\mathcal{A}(B, Y, X)$ and exchange graph $\text{Ex}(B, Y, X)$ where Y are principal coefficients. These depend on X only up to isomorphism.

Recall from Section 2.1 that we associate to B a Cartan matrix A , simple roots Π and simple co-roots Π^\vee . The **fundamental weights** are the vectors in the basis of V^* that is dual to the basis Π^\vee for V . Since the indexing set I indexes rows and columns of B , it also indexes rows and columns of A , and thus indexes Π and Π^\vee . We write ρ_i for the fundamental weight that is dual to α_i^\vee . The **weight lattice** is the lattice generated by the fundamental weights.

In a cluster algebra with principal coefficients, the **g-vector** $\mathbf{g}(x)$ of a cluster variable x is a vector in the weight lattice, defined by the following recursion: We will write \mathbf{g}_e^v for $\mathbf{g}(x_e^v)$. For each $x_i^{v_b} = x_i$ in the initial cluster X , the g-vector of x_i is ρ_i . For other cluster variables, the g-vector is defined by the following recursion.

g-vector mutation. Let e be an edge v to v' . The g-vectors of the cluster $X^{v'}$ are given by

$$(3.6) \quad \mathbf{g}_f^{v'} = \begin{cases} -\mathbf{g}_e^v + \sum_{p \in I(v)} [-b_{pe}^v]_+ \mathbf{g}_p^v - \sum_{i \in I} [-b_{ie}^v]_+ \mathbf{b}_i & \text{if } p = e \\ \mathbf{g}_f^v & \text{if } p \neq e. \end{cases}$$

Here, \mathbf{b}_i is the vector in V^* whose fundamental-weight coordinates are given by the i^{th} column of the initial exchange matrix B .

Usually, the \mathbf{g} -vector is defined as an integer vector rather than a vector in the weight lattice. The integer vector can be recovered by taking fundamental-weight coordinates. Taking the recursive formula above as a definition, it is not immediately clear that the \mathbf{g} vector is well-defined, but the definition in [13, Sections 6–7] does not have this problem. The recursive formulation above is the alternate form [13, Equation (6.13)] of [13, Proposition 6.6], rewritten to define a vector in the weight lattice.

The motivating questions of this paper are how to compute the exchange graph, the denominator vectors, the \mathbf{g} -vectors, and the matrices B^v and H^v . As we will see in this paper, *the matrices H^v should be considered the most fundamental objects.*

3.2. Polyhedral geometry. In discussing cluster algebras and frameworks, it will be useful to use the language of polyhedral cones and fans. We briefly review some background material. A **closed polyhedral cone** is a subset F of V^* that is of the form $\bigcap \{x \in V^* : \langle x, \beta_i \rangle \geq 0\}$ for a finite list of vectors β_1, \dots, β_k in V^* . Equivalently, a closed polyhedral cone is the nonnegative linear span of a finite set of vectors in V^* . In this paper, the term **cone** will be used as a shorthand for “closed polyhedral cone.” The cone is called **simplicial** if β_1, \dots, β_k can be chosen so as to form a basis for V , or equivalently, if it is the nonnegative linear span of a basis for V^* . One can similarly define cones and fans in V , but we will only consider them in V^* .

If F is a cone, then a subset G of F is called a **face** of F if there is some linear functional λ in V^* that is nonnegative on F and 0 on G . Note that F is a face of itself. (Take the zero linear functional.) A **facet** of F is a face G of F with $\dim(G) = \dim(F) - 1$. The **relative interior** of a cone F is the set of points of F not in any proper face of F . Topologically, the relative interior is the interior of F as a subset of $\text{Span}_{\mathbb{R}}(F)$.

We’ll say that cones F_1 and F_2 **meet nicely** if $F_1 \cap F_2$ is a face of both F_1 and F_2 . A collection \mathcal{F} of cones in V^* is called a **fan** if

- (1) For any cone F in \mathcal{F} , and any face G of F , the cone G is also in \mathcal{F} .
- (2) Any two cones F_1 and F_2 in \mathcal{F} meet nicely.

See [7, Chapter V] for more on fans.

We will need some well-known, easy facts from polyhedral geometry:

Proposition 3.2 (Chapter II of [7]). *A cone has finitely many faces, each of which is itself a cone.*

Proposition 3.3 (Proposition 2.3 of [28]). *If F is a cone, G is a face of F , and H is a face of G , then H is a face of F .*

Proposition 3.4 (Proposition 2.3 of [28]). *If F is a cone, then any two faces of F meet nicely.*

Proposition 3.5 (Lemma 14 of [18]). *Suppose F and G are cones that meet nicely. Let F' be a face of F and let G' be a face of G . Then F' and G' meet nicely.*

Proposition 3.5 immediately implies the following lemma, which simplifies the process of checking that a set of cones is a fan.

Lemma 3.6. *Let \mathcal{C} be a collection of cones. Suppose that every pair of cones in \mathcal{C} meet nicely. Let \mathcal{F} be the collection of all faces of cones in \mathcal{C} . Then \mathcal{F} is a fan.*

3.3. Some conjectures about cluster algebras. In this section, we review some conjectures from [12, 13] and make a few new conjectures that are suggested by the results of this paper.

The following conjecture is [12, Conjecture 4.14(3)]. Informally, the conjecture says that, if two cluster variables are equal to each other, then they are equal for an obvious reason.

Conjecture 3.7. *For any cluster variable x , the seeds whose clusters contain x induce a connected subgraph of the exchange graph.*

We will prove the following stronger conjecture for some matrices B .

Conjecture 3.8. *For any set \mathcal{X} of cluster variables, the seeds whose clusters contain \mathcal{X} as a subset induce a connected subgraph of the exchange graph.*

Two extended exchange matrices \tilde{B}^u and \tilde{B}^v are equivalent if there exists a bijection $\lambda : I(u) \rightarrow I(v)$ such that $b_{ef}^u = b_{\lambda(e)\lambda(f)}^v$ for all $e, f \in I(u)$ and $b_{ie}^u = b_{i\lambda(e)}^v$ for all $e \in I(u)$. The following is [13, Conjecture 4.7].

Conjecture 3.9. *Let u and v be vertices in the exchange graph of a cluster algebra with principal coefficients. Then \tilde{B}^u and \tilde{B}^v are equivalent if and only if $u = v$.*

In other words, when we associate a seed to each vertex of the n -regular tree T , the seeds associated to two vertices are equivalent if and only if the two extended exchange matrices are equivalent. We offer the following strengthening of Conjecture 3.9.

Conjecture 3.10. *Let u and v be vertices in the exchange graph of a cluster algebra with principal coefficients. Then H^u and H^v are equivalent if and only if $u = v$.*

The following conjecture is [13, Conjecture 7.10(2)].

Conjecture 3.11. *For each $v \in \text{Ex}_0(B)$, the vectors $\mathbf{g}(x_e^v) : e \in I(v)$ are a \mathbb{Z} -basis for the weight lattice.*

A collection of vectors in \mathbb{R}_n is **sign-coherent** if, for all i from 1 to n , the i^{th} coordinates of the vectors are either all nonnegative, or all nonpositive.

Conjecture 3.12. *For each $v \in \text{Ex}_0(B)$, the vectors $\mathbf{g}(x_e^v) : e \in I(v)$ are sign-coherent.*

A **cluster monomial** is a monomial in the cluster variables contained in a single cluster. The **support** of a cluster monomial is the set of cluster variables appearing in the monomial with nonzero exponent. The **\mathbf{g} -vector** of a cluster monomial is the product (with multiplicities) of the **\mathbf{g} -vectors** of the cluster variables in its support.

The following is [12, Conjecture 4.16].

Conjecture 3.13. *The cluster monomials form a linearly independent set.*

For each vertex v in the exchange graph, let $\text{Cone}(v)$ be the cone in V^* spanned by the weights $\mathbf{g}(x_e^v) : e \in I(v)$. Note that we earlier defined $\text{Cone}(v)$ for v a vertex of a framework, and we have now defined $\text{Cone}(v)$ for v a vertex of the exchange graph. Theorem 3.24(3) will show that these notations are compatible when they both make sense. The following is a restatement of [13, Conjecture 7.10(1)]. (For another restatement, see [6, Conjecture 1.5].)

Conjecture 3.14. *The collection $\{\text{Cone}(v) : v \in \text{Ex}_0(B)\}$ is a fan and the map $v \mapsto \text{Cone}(v)$ is injective.*

In particular (and as stated in [13, Conjecture 7.10(1)]), different cluster monomials have different \mathbf{g} -vectors. Conjectures 3.14 and 3.16 imply Conjecture 3.13, as explained in [13, Remark 7.11].

The assertion that Conjecture 3.8 and Conjecture 3.14 both hold is equivalent to the following conjecture:

Conjecture 3.15. *Suppose two cluster monomials have the same \mathbf{g} -vector. If one is supported on some set \mathcal{X} of cluster variables in a seed, and the other is supported on some set \mathcal{X}' of cluster variables in another seed, then $\mathcal{X} = \mathcal{X}'$, and furthermore, the two seeds are related by a sequence of seed mutations that do not exchange any variables in \mathcal{X} .*

Each cluster variable x is a rational function in the initial cluster variables $x_i : i \in I$ and the initial coefficients $y_i : i \in I$. The F -**polynomial** of x is obtained by specializing each x_i to 1 in this rational function. Theorem 3.1 implies that the F -polynomial is a polynomial in $y_i : i \in I$ with integer coefficients. The following are [13, Conjecture 5.4] and [13, Conjecture 5.5].

Conjecture 3.16. *Each F -polynomial has constant term 1.*

Conjecture 3.17. *Each F -polynomial has a unique monomial of maximal degree. This monomial has coefficient 1 and is divisible by all other monomials in the F -polynomial.*

In light of [13, Proposition 5.3], Conjecture 3.16 holds for all B if and only if and Conjecture 3.17 holds for all B . More specifically, given a particular B , Conjecture 3.16 holds for B and $-B$ if and only if and Conjecture 3.17 holds for B and $-B$.

There are several other formulations of Conjecture 3.16 listed in [13, Proposition 5.6]. We will state one of them, which, we will see, corresponds to the Sign condition. The following condition appears as condition (ii') in the proof of [13, Proposition 5.6], where it is shown to be equivalent to Conjecture 3.16.

Conjecture 3.18. *For each vertex $v \in \text{Ex}_0(B)$, the rows of H^v are sign-coherent.*

We add a new conjecture that is suggested by the Cambrian framework constructed in Section 5. We define a map $\nu : V \rightarrow V^*$ by setting

$$\nu(\alpha_j) = - \sum_{i \in I} E(\alpha_i^\vee, \alpha_j) \rho_i.$$

When B is acyclic, ν is given by the negative of an upper uni-triangular matrix, and therefore it is invertible. The inverse matrix is easily constructed by a standard combinatorial trick. Define a bilinear form F on V by setting

$$F(\alpha_i^\vee, \alpha_j) = \sum (-E(\alpha_{i_0}^\vee, \alpha_{i_1})) (-E(\alpha_{i_1}^\vee, \alpha_{i_2})) \cdots (-E(\alpha_{i_{k-1}}^\vee, \alpha_{i_k})),$$

where the sum is over all paths $i = i_0 \rightarrow i_1 \rightarrow \cdots \rightarrow i_k = j$ in the complete graph with vertices I . If $k = 0$, then the summand is interpreted as 1. Since B is acyclic, this is really a finite sum. Define $\eta : V^* \rightarrow V$ by

$$\eta(\rho_j) = - \sum_{i \in I} F(\alpha_i^\vee, \alpha_j) \alpha_j.$$

It is easy to verify that the maps η and ν are inverse to each other.

Conjecture 3.19. *If B is acyclic and x is a cluster variable not contained in the initial seed, then $\mathbf{g}(x) = \nu(\mathbf{d}(x))$. Equivalently, $\mathbf{d}(x) = \eta(\mathbf{g}(x))$.*

As written, the conjecture relates a vector in the weight lattice to a vector in the root lattice. Equivalently, the conjecture says that the \mathbf{g} -vector and the denominator vector, realized as integer vectors, are related by the action of the matrices associated to $-E$ and $-F$. This conjecture is particularly interesting in connection with [13, Proposition 7.16], [13, Conjecture 7.17], and [13, Conjecture 6.11]. We will see in Theorem 5.29 that the conjecture holds when B is of finite Cartan type. It is also easily verified when $n = 2$.

If Conjecture 3.19 holds, then Conjectures 3.11, 3.14, and 3.15 imply the following three conjectures in the case where B is acyclic. The second of the three is [12, Conjecture 4.17], and can be restated in the language of fans, like Conjecture 3.14. The first is only stated in the case of acyclic B in light of a counterexample to the general conjecture given in [13, Remark 7.7].

Conjecture 3.20. *Suppose B is acyclic. For each $v \in \text{Ex}_0(B)$, the vectors $\mathbf{d}(x_e^v) : e \in I(v)$ are a \mathbb{Z} -basis for the weight lattice.*

Conjecture 3.21. *Different cluster monomials have different denominator vectors.*

Conjecture 3.22. *Conjecture 3.15 holds with \mathbf{g} -vectors replaced by denominator vectors.*

We now review the state of these conjectures before and after the results proved in this paper.

Conjectures 3.7 and 3.8 are proved in [11] for B of finite type. Conjectures 3.16, 3.17 and 3.18 are proved in [13] for bipartite B , for principal coefficients and F -polynomials lying in certain seeds. Conjectures 3.11, 3.13, 3.14, 3.20, 3.21 and 3.22 are also proved in [13] for bipartite B of finite Cartan type. See [13, Section 13] for details. These same conjectures were proved for arbitrary B of finite Cartan type in [27]. Conjecture 3.16 (and thus Conjecture 3.17) is proved for B of finite Cartan type as [27, Corollary 1.13]. Conjectures 3.11, 3.12, 3.14, 3.16, and 3.17, along with another conjecture from [13], are proved in [6], under the assumption that B is skew-symmetric, rather than just skew-symmetrizable. (See also [14].) The constructions of [6] were generalized in [5] to prove the same conjectures for a class of exchange matrices B that includes all acyclic B . Since Conjectures 3.14 and 3.16 hold for B of finite Cartan type, so do Conjectures 3.13 and 3.18, as explained above. Conjecture 3.15 (and therefore Conjecture 3.14) was already proved for B of finite Cartan type, in a somewhat similar manner to the proof in this paper, as explained in Remark 5.14.

In this paper, we use frameworks to give an independent proof of Conjectures 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, 3.18, 3.19, 3.20, 3.21 and 3.22 for B of finite Cartan type. In particular, Conjectures 3.9, 3.10, and 3.19 appear to be new for B of finite Cartan type.

3.4. From frameworks to cluster algebras. We now show that every framework is a combinatorial model for the associated cluster algebra.

Let G be a quasi-graph. A **covering** of G is a quasi-graph G' and a surjective map $p : G' \rightarrow G$ such that if v and v' are connected by an edge in G' , then $p(v)$ and

$p(v')$ are connected by an edge in G , and such that p induces a bijection between full edges incident to v and full edges incident to $p(v)$. We also require that half-edges of v and half-edges of $p(v)$ are in bijection and require a covering map p to include a specific choice of bijection from half-edges of each $v \in G'$ to half-edges of $p(v)$.

Given an n -regular, connected quasi-graph G and a vertex v_b of G , we define the **universal cover** \hat{G} of G at v_b and a corresponding covering map p . The vertices of \hat{G} are sequences u_0, \dots, u_k for $k \geq 0$ such that $u_0 = v_b$ and each pair u_{i-1}, u_i is connected by an edge e_i of G , subject to the restriction that $e_i \neq e_{i+1}$ for i from 1 to $n-1$. If $k > 0$, then u_0, \dots, u_k is connected to u_0, \dots, u_{k-1} by an edge in \hat{G} , and these are all of the full edges of \hat{G} . Also, for each half-edge incident to v_k in G , there is exactly one half-edge incident to u_0, \dots, u_k in \hat{G} , and these are all of the half-edges of \hat{G} . The covering map $p : \hat{G} \rightarrow G$ is the map sending u_0, \dots, u_k to u_k . For each vertex u_0, \dots, u_k of \hat{G} , we fix any bijection between the half-edges incident to u_0, \dots, u_k and the half-edges incident to u_k , and use these bijections to complete the definition of p . Thus $p : \hat{G} \rightarrow G$ is a covering. The universal cover \hat{G} has no cycles, and p is the identity map on vertices if and only if G has no cycles.

A framework (G, C, C^\vee) lifts to a framework $(\hat{G}, \hat{C}, \hat{C}^\vee)$ by setting $\hat{C}(v, e) = C(p(v), p(e))$ and $\hat{C}^\vee(v, e) = C^\vee(p(v), p(e))$. We will overload notation by dropping the hats from \hat{C} and \hat{C}^\vee and we will refer to (\hat{G}, C, C^\vee) as the universal cover of (G, C, C^\vee) . The maps $\mu_e : I(v) \rightarrow I(v')$ also lift to \hat{G} in the obvious way.

Theorem 3.23. *Suppose (G, C, C^\vee) is a framework for B and let $\mathcal{A} = \mathcal{A}(B, Y, X)$ be a cluster algebra whose initial exchange matrix is B . Then there exists a covering $v \mapsto \text{Seed}(v) = (B^v, Y^v, X^v)$ from the universal cover \hat{G} to the exchange graph $\text{Ex}(B, Y, X)$ of \mathcal{A} , sending v_b to the initial seed, such that the exchange matrix $B^v = [b_{ef}^v]_{e,f \in I(v)}$ has $b_{ef}^v = \omega(C^\vee(v, e), C(v, f))$.*

The assertion that $\text{Seed} : \hat{G} \rightarrow \text{Ex}(B, Y, X)$ is a covering means that this labeling of rows and columns of the exchange matrix by edges in \hat{G} makes sense.

Theorem 3.24. *Suppose (G, C, C^\vee) is a framework for B and let $\mathcal{A}_0 = \mathcal{A}_0(B)$ be a cluster algebra with principal coefficients whose initial exchange matrix is B . Let v be a vertex of \hat{G} . Then*

- (1) *If $H^v = [h_{ie}^v]_{i \in I, e \in I(v)}$ is the bottom half of the extended exchange matrix \tilde{B}^v associated to $\text{Seed}(v)$, then h_{ie}^v is $[\alpha_i : C(v, e)]$, the coefficient of α_i in the simple root expansion of $C(v, e)$.*
- (2) *The rows of H^v are sign-coherent.*
- (3) *If $X^v = (x_e^v : e \in I(v))$ is the cluster in $\text{Seed}(v)$, then for each $e \in I(v)$, the \mathbf{g} -vector $\mathbf{g}_e^v = \mathbf{g}(x_e^v)$ is $R(v, e)$.*
- (4) *The \mathbf{g} -vectors $(\mathbf{g}_e^v : e \in I(v))$ are a basis for the weight lattice.*
- (5) *If F_e^v is the F -polynomial of x_e^v , then the constant term of F_e^v is 1.*

We first prove Theorem 3.23. For each vertex of G , define $B^v = [b_{ef}^v]_{e,f \in I(v)}$ by setting $b_{ef}^v = \omega(C^\vee(v, e), C(v, f))$. The matrix B^{v_b} coincides with the initial exchange matrix B , where $I(v_b)$ is identified with I as explained in connection with the Base condition in Section 2.1.

Lemma 3.25. *Suppose (G, C, C^\vee) is a framework for B . If v and v' are vertices in G , connected by the edge e , then the matrices B^v and $B^{v'}$ are related by matrix mutation at e , and by applying μ_e to the row and column indices.*

Proof. Let e, e' and e'' be distinct edges in $I(v)$. Let $\beta = C(v, e)$, $\gamma = C(v, e')$ and $\delta = C(v, e'')$ and let $\beta' = C(v', \mu_e(e))$, $\gamma' = C(v', \mu_e(e'))$ and $\delta' = C(v', \mu_e(e''))$. The corresponding co-labels will be denoted by adding $^\vee$ to β , etc. Since matrix mutation is an involution, and by the Transition condition and the Co-label condition, we may as well take $\text{sgn}(\beta) = \text{sgn}(\beta^\vee) = 1$. The proof consists of verifying the following three identities.

$$(3.7) \quad \omega((\beta')^\vee, \gamma') = -\omega(\beta^\vee, \gamma)$$

$$(3.8) \quad \omega((\gamma')^\vee, \beta') = -\omega(\gamma^\vee, \beta)$$

$$(3.9) \quad \omega((\gamma')^\vee, \delta') = \omega(\gamma^\vee, \delta) + \text{sgn}(\omega(\gamma^\vee, \beta)) [\omega(\gamma^\vee, \beta), \omega(\beta^\vee, \delta)]_+$$

The Transition condition, with $\text{sgn}(\beta) = 1$, says that $\gamma' = \gamma + [\omega(\beta^\vee, \gamma)]_+ \beta$ and similarly the Co-transition condition says that $(\gamma')^\vee = \gamma^\vee + [-\omega(\gamma^\vee, \beta)]_+ \beta^\vee$. Thus (3.7) and (3.8) follow immediately from the linearity of ω and the fact that $\omega(\beta^\vee, \beta) = 0$. Again using the Transition condition, $\delta' = \delta + [\text{sgn}(\beta)\omega(\beta^\vee, \delta)]_+ \beta$, so

$$\omega((\gamma')^\vee, \delta') = \omega(\gamma^\vee, \delta) + \omega(\beta^\vee, \delta)[- \omega(\gamma^\vee, \beta)]_+ + \omega(\gamma^\vee, \beta)[\omega(\beta^\vee, \delta)]_+.$$

It is now trivial to check that (3.9) holds in all four cases given by $\pm\omega(\beta^\vee, \delta) \geq 0$ and $\pm\omega(\gamma^\vee, \beta) \leq 0$. \square

Lemma 3.25 has an immediate extension: In the statement of the lemma, we may replace (G, C, C^\vee) by (\hat{G}, C, C^\vee) .

We next extend $v \mapsto B^v$ to a map from vertices of \hat{G} to seeds. The seeds are defined recursively in terms of completely general coefficients. The initial seed is given by the following data: the exchange matrix $B = B^{v_b}$, an n -tuple $Y = Y^{v_b} = (y_i : i \in I)$ of elements of the semifield \mathbb{P} , and an n -tuple $X = X^{v_b} = (x_i : i \in I)$ of elements of the field \mathcal{F} . We continue to identify I with $I(v_b)$.

We define, for each vertex v of \hat{G} , a tuple $Y^v = (y_e^v : e \in I(v))$ and a tuple $X^v = (x_e^v : e \in I(v))$ and set $\text{Seed}(v) = (B^v, Y^v, X^v)$. Recall a vertex of \hat{G} is a path χ in G , starting at v_b , such that no edge occurs twice consecutively in the path. Given such a path χ , representing a vertex v of \hat{G} , we define $\text{Seed}(v)$ in the obvious way: If χ is a single vertex (necessarily v_b), then $\text{Seed}(v)$ is the initial seed. Otherwise, let e be the last edge in the path and let χ' be the path in G obtained from χ by deleting the last vertex of the path χ . Let v' be the vertex of \hat{G} corresponding to χ' . Then $\text{Seed}(v')$ is defined by induction, and we define $\text{Seed}(v)$ to be the seed obtained by mutating $\text{Seed}(v')$ at the edge e .

We have constructed the map Seed so that it has the property that, for adjacent vertices v and v' in \hat{G} , connected by the edge e , the seeds $\text{Seed}(v)$ and $\text{Seed}(v')$ are related by mutation at e , and by applying μ_e to the indexing sets. Together with Lemma 3.25, this completes the proof of Theorem 3.23.

Now suppose, $\mathcal{A} = \mathcal{A}_0(B)$. For each vertex of G , define $\tilde{B}^v = \begin{pmatrix} B^v \\ H^v \end{pmatrix}$, where B^v is the matrix defined before Lemma 3.25 and $H^v = [h_{ie}^v]_{i \in I, e \in I(v)}$ is defined by setting $h_{ie}^v = [\alpha_i : C(v, e)]$. Then H^{v_b} is the identity matrix by the Base condition.

Lemma 3.26. *Suppose (G, C, C^\vee) is a framework for B . Let v and v' be adjacent in G , via the edge e . Then \tilde{B}^v and $\tilde{B}^{v'}$ are related by matrix mutation at e , and by applying μ_e to the column indices and to the indices $I(v)$ of the top n rows.*

Proof. Lemma 3.25 establishes that the restrictions to top square submatrices are indeed related by matrix mutation. The remainder of the proof consists of verifying the following two identities, for $i \in I$ and for β, β', γ , and γ' as in the proof of Lemma 3.25. As in that proof, we will also assume that $\text{sgn}(\beta) = 1$.

$$(3.10) \quad [\alpha_i : \beta'] = -[\alpha_i : \beta]$$

$$(3.11) \quad [\alpha_i : \gamma'] = [\alpha_i : \gamma] + \text{sgn}([\alpha_i : \beta]) [[\alpha_i : \beta] \omega(\beta^\vee, \gamma)]_+$$

The identity (3.10) is immediate by the Transition condition. Since $\text{sgn}(\beta) = 1$, the right side of (3.11) is $[\alpha_i : \gamma] + [\alpha_i : \beta] [\omega(\beta^\vee, \gamma)]_+$, which, by the Transition condition, is the coefficient of α_i in $\gamma' = \gamma + [\omega(\beta^\vee, \gamma)]_+ \beta$. \square

Lemma 3.26 proves Theorem 3.24(1), and Theorem 3.24(2) follows immediately by the Sign condition.

Proof of Theorem 3.24(3). When $v = v_b$, the vector $R(v_b, i)$ is the fundamental weight dual to α_i^\vee , for each $i \in I = I(v_b)$, so the assertion holds in this case. Thus by Theorem 3.23, we need only check that the assertion at a vertex v implies the assertion at any adjacent vertex v' .

Let v and v' be connected by an edge e . Without loss of generality, take $\text{sgn}(C(v, e)) = 1$. Using Theorem 3.23 and assuming the assertion for \mathbf{g} -vectors at v , the \mathbf{g} -vector recursion (3.6) says that $\mathbf{g}_{\mu_e(f)}^{v'} = R(v, \mu_e(f))$ if $f \in I(v') \setminus \{e\}$, and that $\mathbf{g}_e^{v'}$ equals

$$(3.12) \quad -R(v, e) - \sum_{p \in I(v)} [\omega(C^\vee(v, p), C(v, e))]_- R(v, p) + \sum_{i \in I} [[\alpha_i : C(v, e)]]_- b_i.$$

By Proposition 2.7, $R(v, \mu_e(f)) = R(v', f)$. It remains to show that (3.12) is $R(v', e)$ by showing that for every $f \in I(v')$, the pairing of $C^\vee(v', f)$ with (3.12) equals δ_{fe} . Since the sign of $C(v, e)$ is 1, the second sum in (3.12) vanishes. In the first sum, $\omega(C^\vee(v, e), C(v, f))$ vanishes for $p = e$ by the antisymmetry of ω . Thus (3.12) is

$$(3.13) \quad -R(v, e) - \sum_{p \in I(v) \setminus \{e\}} [\omega(C^\vee(v, p), C(v, e))]_- R(v, p)$$

The Co-transition condition says that $C^\vee(v', e) = -C^\vee(v, e)$, so the pairing of (3.13) with $C^\vee(v', e)$ is $\langle -R(v, e), -C^\vee(v, e) \rangle = 1$. Suppose $f \in I(v') \setminus \{e\}$. Since the sign of $C(v, e)$ is 1, the Co-transition condition says that $C^\vee(v', f)$ equals $C^\vee(v, \mu_e(f)) - [\omega(C^\vee(v, \mu_e(f)), C(v, e))]_- C^\vee(v, e)$. Thus (3.13) paired with $C^\vee(v', f)$ is

$$[\omega(C^\vee(v, \mu_e(f)), C(v, e))]_- - [\omega(C^\vee(v, \mu_e(f)), C(v, e))]_- = 0. \quad \square$$

Theorem 3.24(4) follows from Theorem 3.24(3) and Proposition 2.3 because the weight lattice is the dual lattice to the co-root lattice. Theorem 3.24(5) follows from Theorem 3.24(2) by the argument given in the proof of [13, Proposition 5.6], which shows that Conjectures 3.16 and 3.18 are equivalent. We must be careful, because [13, Proposition 5.6] states that one conjecture, for all B , is equivalent to the other conjecture, for all B . Theorems 3.24(5) and Theorem 3.24(2) refer to a specific B , and perhaps only to part of the exchange graph. However, the proof of [13, Proposition 5.6] does not require B to vary, and argues by induction on distance, in the exchange graph, to the initial seed. Since G is connected by hypothesis, the argument goes through. This completes the proof of Theorem 3.24.

3.5. From cluster algebras to frameworks. We now show that, assuming Conjecture 3.18, every exchange matrix B has a framework. The point is to validate the notion of a framework by showing that, if cluster algebras behave as we expect them to, frameworks are unavoidable. In the process, we establish some interesting statements about cluster algebras.

Let B be an exchange matrix with Cartan companion A . Let T be the n -regular tree considered in Section 3.1 and recall the maps μ_e defined for each edge e . Let $v \mapsto \tilde{B}^v$ be the map that associates to each vertex of T an extended exchange matrix, with exchange matrix B and principal coefficients at the base vertex. Define a labeling H of incident pairs in T by taking $H(v, e)$ to be the vector whose simple root coordinates are given by the bottom n entries of the column of \tilde{B}^v labeled by e . Similarly, we consider a different map $v \mapsto (\tilde{B}')^v$ that associates to each vertex of T an extended exchange matrix, with exchange matrix $-B^T$ and principal coefficients at the base vertex. Define a co-labeling H^\vee by taking $H^\vee(v, e)$ to be the vector whose simple co-root coordinates are given by the bottom n entries of the column of $(\tilde{B}')^v$ labeled by e . (Recall from the discussion immediately following Proposition 2.9 that the simple co-roots associated to A are the simple roots associated to A^T , the Cartan companion of $-B^T$.)

Theorem 3.27. *Suppose Conjecture 3.18 holds for B . Then the triple (T, H, H^\vee) is a framework for B .*

Proof. The hypothesis that Conjecture 3.18 holds for B is exactly the statement that the Sign condition holds. Consider the following conditions:

- (i) $H^\vee(v, e)$ is a positive scalar multiple of $H(v, e)$ for all $e \in I(v)$.
- (ii) The exchange matrix B^v has entries $\omega(H^\vee(v, e), H(v, f))$ for $e, f \in I(v)$.
- (iii) The exchange matrix $(B')^v$ has entries $\omega(H(v, e), H^\vee(v, f))$ for $e, f \in I(v)$.

These conditions hold at the base vertex. Suppose now they hold at a vertex v and suppose that v' is an adjacent vertex. Given that (ii) holds at v , the proof of Lemma 3.26 establishes that the (strengthened) Transition condition holds for v and v' . Similarly, given that (iii) holds at v , the proof of Lemma 3.26 establishes that the (strengthened) Co-transition condition holds for v and v' . Now the proof of Proposition 2.5 is easily modified to show that condition (i) holds at v' as well. Thus (i) holds at every vertex, or in other words, the Co-label condition holds. We have also established the Transition and Co-transition conditions, and the Base condition is immediate. \square

We establish several corollaries. First, combining Theorems 3.24 and 3.27, we obtain the following result, which is the first assertion of [19, Theorem 1.2]. Let v be any vertex in T . Let G^v be the matrix whose rows are the fundamental weight coordinates of the \mathbf{g} -vectors \mathbf{g}_e^v for $e \in I(v)$. As before, H^v is the bottom half of \tilde{B}^v , the principal-coefficients extended exchange matrix at v .

Corollary 3.28. *Suppose Conjecture 3.18 holds for B and let v be any vertex of T . Then the matrices G^v and H^v are inverse to each other.*

Naturally, the corollary assumes that we have chosen the same linear order on $I(v)$ to write both matrices.

Corollary 3.29. *Suppose Conjecture 3.18 holds for B and suppose Conjecture 3.14 holds for B and for $-B^T$. Then the fans associated to B and $-B^T$ coincide.*

Proof. By Theorem 3.24(3) and Theorem 3.27, the cones of the fan associated to B are exactly the cones defined by the label sets $H(v)$ for vertices v of T . But Proposition 2.9, together with the same two theorems, implies that the cones of the fan associated to $-B^T$ are exactly the cones defined by the label sets $H^\vee(v)$ for vertices v of T . The Co-label condition says that these are the same cones. \square

Corollary 3.30. *Suppose Conjectures 3.14 and 3.18 hold for B . Then Conjecture 3.10 holds for B .*

Proof. Suppose u and v are vertices of T with $H^u = H^v$. Then Theorem 3.24(3) and Theorem 3.27 imply that $\text{Cone}(u) = \text{Cone}(v)$. Now Conjecture 3.14 implies that $u = v$. \square

Recall that $\text{Ex}_0(B)$ is the exchange graph of the principal-coefficients cluster algebra associated to B .

Corollary 3.31. *Suppose Conjecture 3.18 holds for B . Suppose either that B is skew-symmetric or that Conjecture 3.14 holds for B and $-B^T$. Then the triple $(\text{Ex}_0(B), H, H^\vee)$ is a framework for B .*

Proof. We first show that $\text{Ex}_0(B)$ and $\text{Ex}_0(-B^T)$ are identical as quotients of the n -regular graph T . This is a tautology if B is skew-symmetric, so we need only consider the case where Conjecture 3.14 holds for B and $-B^T$. We need to show that, for any two vertices u and v of T , we have \tilde{B}^u equivalent to \tilde{B}^v if and only if $(\tilde{B}')^u$ is equivalent to $(\tilde{B}')^v$. By symmetry, we need only check one direction. Suppose \tilde{B}^u is equivalent to \tilde{B}^v , so that the cones defined by $H(u)$ and $H(v)$ coincide. Then the Co-label condition implies that the cones defined by $H^\vee(u)$ and $H^\vee(v)$ coincide. Since Conjecture 3.14 holds for $-B^T$, we conclude that $(\tilde{B}')^u$ is equivalent to $(\tilde{B}')^v$.

Since $\text{Ex}_0(B)$ and $\text{Ex}_0(-B^T)$ are identical as quotients, the quotient inherits the labeling H and the co-labeling H^\vee . The triple $(\text{Ex}_0(B), H, H^\vee)$ inherits the Co-label, Sign, Base, Transition, and Co-transition conditions from (T, H, H^\vee) . \square

4. GLOBAL CONDITIONS ON FRAMEWORKS

All of the conditions defining a framework are local. In this section, we consider some global conditions on a framework and show how the existence of a framework for B with various global properties establishes, for B , various conjectures from Section 3.3. All the results in this section apply to general frameworks, whether or not they are reflection frameworks.

4.1. Complete, exact, and well-connected frameworks. We say that a framework (G, C, C^\vee) is **complete** if G has no half-edges. This is a local condition, but it is convenient to discuss it together with the other global conditions discussed in this section. The universal cover of a complete framework is the n -regular tree, and thus it is easily seen that the map Seed must be surjective when the framework is complete. Thus Theorem 3.24 implies the following theorem.

Theorem 4.1. *If a complete framework exists for B , then Conjectures 3.11, 3.16, and 3.18 all hold for B . If in addition a complete framework exists for $-B$, then Conjecture 3.17 also holds for B .*

The assertion about Conjecture 3.17 holds by the relationship between Conjectures 3.16 and 3.17 explained in Section 3.3.

A framework is **injective** if, for every pair u, v of vertices of G , the following three conditions are equivalent: $C(u) = C(v)$; $C^\vee(u) = C^\vee(v)$; and $u = v$.

Consider the special case of Theorem 3.23 where (B, Y, X) is a seed with principal coefficients, so that $\mathcal{A}(B, Y, X)$ is $\mathcal{A}_0(B)$. Given a framework (G, C, C^\vee) , Theorem 3.23 asserts the existence of a map $v \mapsto \text{Seed}(v)$ from the universal cover \hat{G} to $\text{Ex}_0(B)$. The framework (G, C, C^\vee) is **ample** if Seed factors through the covering map $\hat{G} \rightarrow G$. A framework is **exact** if it is injective and ample. Note that exactness and ampleness are not purely combinatorial, but depend on the framework's interaction with a cluster algebra.

Theorem 4.2. *Suppose (G, C, C^\vee) is an exact framework for B and let $\mathcal{A}_0(B)$ be the cluster algebra with principal coefficients whose initial exchange matrix is B . Then the map $v \mapsto \text{Seed}(v) = (\tilde{B}^v, X^v)$ is a graph isomorphism from G (ignoring half-edges) to its image, a subgraph of the exchange graph $\text{Ex}_0(B)$. If (G, C, C^\vee) is also complete, then the map is a graph isomorphism from G to $\text{Ex}_0(B)$.*

Proof. Since (G, C, C^\vee) is ample, the map $v \mapsto \text{Seed}(v)$ descends to a map $v \mapsto \text{Seed}(v)$ from G to $\text{Ex}_0(B)$. We will show that the map is an isomorphism to its image. If u and v are vertices of G with $\text{Seed}(u) = \text{Seed}(v)$, then Theorem 3.24(1) implies that $C(u) = C(v)$. Since (G, C, C^\vee) is an injective framework, we conclude that $v \mapsto \text{Seed}(v)$ is one-to-one. Now G is n -regular and Seed maps the neighbors of each vertex v to distinct neighbors of $\text{Seed}(v)$ in $\text{Ex}_0(B)$. Since $\text{Ex}_0(B)$ is also n -regular, an easy proof by induction on the distance from the initial seed shows that $v \mapsto \text{Seed}(v)$ is an isomorphism to its image. If G has no half-edges, then the image of Seed is an n -regular subgraph of the n -regular graph $\text{Ex}_0(B)$. Since $\text{Ex}_0(B)$ is connected, the image is all of $\text{Ex}_0(B)$. \square

Remark 4.3. Theorem 4.2 implies that, up to isomorphism of G , there is at most one complete, exact framework for a given B . Furthermore, non-complete, exact frameworks coincide where they overlap, in a sense that can be made precise. However, in this circumstance, the phrase “up to isomorphism” allows some meaningful freedom. Making a useful framework means choosing an appropriate combinatorial, algebraic, or geometric realization of the triple (G, C, C^\vee) .

Theorem 4.2 combines with Theorems 3.24(1) to give the following corollary.

Corollary 4.4. *If a complete, exact framework exists for B , then Conjecture 3.10 holds for B .*

The framework (G, C, C^\vee) is **polyhedral** if the cones $\text{Cone}(v)$, where v ranges over all vertices of G , are the maximal cones of a fan with the property that distinct vertices v of G define distinct cones $\text{Cone}(v)$. The fan in question is always composed of simplicial cones. A polyhedral framework is **well-connected** if it has the following property: If F is a face of $\text{Cone}(v)$ and of $\text{Cone}(v')$, then there exists a path $v = v_0, v_1, \dots, v_k = v'$ in G such that F is a face of $\text{Cone}(v_i)$ for all i from 0 to k . The following theorem is the reason for considering well-connected polyhedral frameworks. It is immediate from Theorem 3.24(3).

Theorem 4.5. *If (G, C, C^\vee) is a polyhedral framework, then \mathbf{g} -vector cones for seeds in the image of $v \mapsto \text{Seed}(v)$ form a fan, identical to the fan defined by*

(G, C, C^\vee) . If (G, C, C^\vee) is also well-connected, then Conjecture 3.15 holds for pairs of cluster monomials supported on clusters in the image of $v \mapsto \text{Seed}(v)$.

Corollary 4.6. *If a complete, exact, well-connected polyhedral framework exists for B , then Conjectures 3.7, 3.8, 3.13, 3.14, and 3.15 all hold for B . Furthermore, the fan defined by the framework is identical to the fan defined by \mathbf{g} -vectors of clusters in $\mathcal{A}_0(B)$.*

Theorem 3.24(3) also implies that a polyhedral framework is automatically injective. Thus a polyhedral framework is exact if and only if it is ample.

Proof. Conjecture 3.15 follows from Theorem 4.5 because the completeness of the framework implies that the image of Seed is the whole exchange graph. As discussed in Section 3.3, Conjecture 3.15 implies the other conjectures except for Conjecture 3.13. Conjecture 3.16 holds by Theorem 4.1. Conjectures 3.14 and 3.16 imply Conjecture 3.13, as explained in Section 3.3. \square

We can expand on Corollary 3.31 to include global conditions.

Theorem 4.7. *Suppose Conjecture 3.18 holds for B . If Conjecture 3.14 holds for B and $-B^T$, then the triple $(\text{Ex}_0(B), H, H^\vee)$ is a complete, exact polyhedral framework for B . It is well-connected if and only if Conjecture 3.15 holds for B .*

Proof. By Corollary 3.31, $(\text{Ex}_0(B), H, H^\vee)$ is a framework, and it is complete because the exchange graph has no half-edges. Conjecture 3.14 for B and $-B^T$ and Theorem 3.24 imply Conjecture 3.10 for B and $-B^T$. Thus $(\text{Ex}_0(B), H, H^\vee)$ is injective. It is ample because the map $\text{Seed} : \widehat{\text{Ex}}_0(B)$ descends to the identity map on $\text{Ex}_0(B)$. The statement about well-connectedness is immediate. \square

Combining Theorem 3.27 and/or Corollary 3.31 and Theorem 4.7 with Theorem 4.1 and Corollaries 4.4 and 4.6, we obtain some additional dependencies among conjectures.

Ampleness is a difficult condition to establish. The easiest way is to know Conjecture 3.10 or 3.14 in advance:

Proposition 4.8. *Suppose (G, C, C^\vee) is a framework for B . If Conjecture 3.10 or Conjecture 3.14 holds for B , then (G, C, C^\vee) is ample.*

Proof. Consider the map Seed from \hat{G} to the principal-coefficients exchange graph $\text{Ex}_0(B)$. If v is the vertex (v_0, \dots, v_k) of \hat{G} , then Theorem 3.24(1) implies that the extended exchange matrix \tilde{B}^v depends only on v_k . Thus if Conjecture 3.10 holds for B , then $\text{Seed}(v)$ depends only on v_k . Furthermore, Theorem 3.24(3) implies that \mathbf{g} -vectors of the cluster X^v depend only on v_k . Thus if Conjecture 3.14 holds for B , then $\text{Seed}(v)$ depends only on v_k . \square

The following sections discuss other ways to prove ampleness.

4.2. Simply connected frameworks. We now define the notion of a simply connected framework, and show that simple connectivity implies ampleness. The definition requires much preparation, beginning with the definition of a *rank-two cycle*.

Suppose (G, C, C^\vee) is a framework, let v be a vertex of G and let e and f be edges incident to v . Construct a doubly infinite sequence of vertices and edges as follows: Set $v_0 = v$, $e_0 = f$ and $e_1 = e$. Then, recursively, let $e_{k+1} = \mu_{e_k}(e_{k-1})$ and

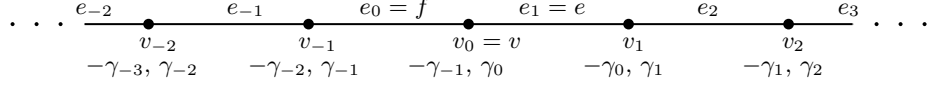


FIGURE 2. The path with vertices v_i , and the relevant elements of each $C(v_i)$

let e_k join v_{k-1} and v_k , as shown in Figure 2. Assume, for the rest of this section, that none of the e_k are half edges, so that this recursion is well-defined.

If the set of distinct edges in this sequence is finite, then the edges define a cycle in G which we will call a **rank-two cycle**.

Let $\gamma_k = C(v_k, e_{k+1})$ and define γ_k^\vee similarly. The Transition and Co-transition conditions imply that $C(v_k, e_k) = -\gamma_{k-1}$, that $C^\vee(v_k, e_k) = -\gamma_{k-1}^\vee$, and that

$$(4.1) \quad \begin{aligned} \gamma_{k+1} &= -\gamma_{k-1} - [\text{sgn}(\gamma_k)\omega(\gamma_k^\vee, \gamma_{k-1})] - \gamma_k \\ \gamma_{k+1}^\vee &= -\gamma_{k-1}^\vee + [\text{sgn}(\gamma_k^\vee)\omega(\gamma_{k-1}^\vee, \gamma_k)] + \gamma_k^\vee \end{aligned}$$

Lemma 4.9. *Let (G, C, C^\vee) be a framework, let v be a vertex and e and f two edges adjacent to v as above. Define v_k , e_k and γ_k as above.*

Suppose that v_k is periodic with some finite period. Then there is some k such that $\text{sgn}(\gamma_k) = \text{sgn}(-\gamma_{k-1}) = -1$.

Proof. Since v_k is periodic, the sequence of cones $\text{Cone}(v_k)$ is periodic. By Corollary 2.8, all of the cones $\text{Cone}(v_k)$ share a common face F of codimension 2 and wind in cyclic order around F . Let U_k be the cone consisting of points $x \in V^*$ with $\langle x, \gamma_k^\vee \rangle \geq 0$ and $\langle x, -\gamma_{k-1}^\vee \rangle \geq 0$. So the codimension-1 faces of U_k both contain F .

Since the $\text{Cone}(v_k)$ repeat periodically, the U_k do as well, winding cyclically around $\text{Span}_{\mathbb{R}}(F)$. So $\bigcup U_k = V^*$. Moreover, all of the vectors γ_k^\vee satisfy the Sign condition, so none of the hyperplanes γ_k^\perp intersect the interior of the negative fundamental domain $-D$. Putting these observations together, there is some k such that $-D \subset U_k$. Since $\langle \gamma, \cdot \rangle$ is positive on $-D$ if and only if $\text{sgn}(\gamma) = -1$, we see that $\text{sgn}(\gamma_k) = \text{sgn}(-\gamma_{k-1}) = -1$ for this k . \square

From equations (4.1) we see that

$$\omega(\gamma_{k-1}^\vee, \gamma_k) = -\omega(\gamma_{k+1}^\vee, \gamma_k) \text{ and } \omega(\gamma_k^\vee, \gamma_{k-1}) = -\omega(\gamma_k^\vee, \gamma_{k+1}).$$

So, if $b = \omega(\gamma_{-1}^\vee, \gamma_0)$ and $c = \omega(\gamma_0^\vee, \gamma_{-1})$, then the values of $\omega(\gamma_k^\vee, \gamma_{k+1})$ alternate $b, -c, b, -c$ et cetera, and the values of $\omega(\gamma_{k+1}^\vee, \gamma_k)$ alternate $c, -b, c, -b$. Note that b and c have opposite signs, and one is 0 if and only if the other is. Switching e and f if necessary, we may assume without loss of generality that $b \geq 0$ and $c \leq 0$.

Motivated by Lemma 4.9, we assume, for the moment, that $\text{sgn}(C(v, e)) = \text{sgn}(C(v, f)) = -1$. Recall that $C(v, e) = \gamma_0$ and $C(v, f) = -\gamma_{-1}$. Let \tilde{V} be the two dimensional vector space spanned by γ_{-1} and γ_0 . Let \tilde{A} be the Cartan matrix $\begin{pmatrix} 2 & c \\ -b & 2 \end{pmatrix}$. Let $\tilde{\Phi}$ be the root system in \tilde{V} defined by \tilde{A} with negative simple roots $-\gamma_{-1}$ and γ_0 , and negative simple co-roots $-\gamma_{-1}^\vee$ and γ_0^\vee . This defines a symmetric bilinear form \tilde{K} on \tilde{V} with $\tilde{K}(\gamma_{-1}^\vee, \gamma_0) = b$ and $\tilde{K}(\gamma_0^\vee, \gamma_{-1}) = -c$.

Lemma 4.10. *Suppose that $C(v, e)$ and $C(v, f)$ are negative roots and make the definitions of the above paragraph, including assuming that $b \geq 0$ and $c \leq 0$ as*

discussed above. Then the vectors $\{\gamma_k\}_{k \in \mathbb{Z}}$ are the almost positive roots (see Section 2.2) of $\tilde{\Phi}$, and $\{\gamma_k^\vee\}_{k \in \mathbb{Z}}$ are the almost positive co-roots, occurring in cyclic order. More specifically, γ_0 and $\gamma_1 = -\gamma_{-1}$ are the negative simple roots of $\tilde{\Phi}$.

Remark 4.11. If $b \leq 0$ and $c \geq 0$, then $\{\gamma_k\}_{k \in \mathbb{Z}}$ is the negatives of the almost positive roots, with γ_{-2} and γ_{-1} being the positive simple roots.

Proof Sketch. This is a straightforward computation, using the recursions (4.1). We have $\gamma_1 = -\gamma_{-1}$ and then

$$\gamma_{k+1} = -\gamma_{k-1} + (b \text{ or } (-c))\gamma_k$$

as long as γ_k is a positive root, where the coefficient of γ_k alternates between b and $-c$ based on the parity of k . This recursion marches through the positive roots of $\tilde{\Phi}$ in order. If the matrix $\begin{pmatrix} 2 & c \\ -b & 2 \end{pmatrix}$ is of infinite type, then the γ_k will always be positive roots. If this matrix is of finite type, the recursion takes on all the positive roots, then becomes γ_0 and γ_1 again, and then repeats the positive roots, indefinitely. So, for $k > 0$, the recursion travels through the almost positive roots in order.

A similar analysis applies for $k < 0$. □

Corollary 4.12. *Let (G, C, C^\vee) be a framework. Let v be a vertex of G with adjacent vertices e and f ; define v_k, e_k, γ_k, b and c as above. Suppose that the v_k form a finite cycle, of length ℓ .*

Then $\begin{pmatrix} 2 & -|c| \\ -|b| & 2 \end{pmatrix}$ is a Cartan matrix of finite type. Moreover, ℓ is divisible by $h + 2$, where h is the Coxeter number of this Cartan matrix.

Proof. By Lemma 4.9, we can reindex the cycle of v_k 's so that γ_k and $-\gamma_{k-1}$ are negative roots. Switching e and f if necessary, we may assume that $b \geq 0$ and $c \leq 0$. By Lemma 4.10, the γ_k go through the almost positive roots of $\tilde{\Phi}$ in circular order. So, since v_k repeats, this shows that the root system $\tilde{\Phi}$ is finite, so $\begin{pmatrix} 2 & c \\ -b & 2 \end{pmatrix}$ is of finite type. □

Remark 4.13. We take the opportunity to remind the reader that $\begin{pmatrix} 2 & -|c| \\ -|b| & 2 \end{pmatrix}$ is of finite type if and only if $|bc| < 4$, and that h is 2, 3, 4 or 6 according to whether $|bc|$ is 0, 1, 2 or 3.

We now prove the key lemmas of this section.

Lemma 4.14. *Let (G, C, C^\vee) be a framework. Let v be a vertex of G with adjacent vertices e and f and define v_k, e_k as above. Suppose that the v_k form a finite cycle of length m . Let $\tilde{v}_0, \tilde{v}_1, \dots$ be an infinite path in \hat{G} lying above the infinite path v_0, v_1, \dots in G . Then $\text{Seed}(\tilde{v}_0) = \text{Seed}(\tilde{v}_m)$.*

Proof. Using Lemma 3.25 to translate [10, Theorem 6.2 and 7.7] into the language of this paper, we obtain the statement that the sequence $\text{Seed}(\tilde{v}_k)$ is periodic with period $h + 2$, where h is the Coxeter number appearing in Corollary 4.12. This, combined with Corollary 4.12 implies the lemma. □

Lemma 4.15. *Let (G, C, C^\vee) be a framework. Let v be a vertex of G with adjacent vertices e and f with $C(v, e)$ and $C(v, f)$ negative. Define v_k, b and c as above.*

Suppose that $\begin{pmatrix} 2 & -|c| \\ -|b| & 2 \end{pmatrix}$ is of finite type, with Coxeter number h . Then $C(v_k)$ and $C^\vee(v_k)$ repeat with period $h + 2$.

Proof. We have already seen that the γ_k repeat in this manner. Let δ be an element of $C(v_0)$ other than γ_0 and $-\gamma_{-1}$. Let δ_k be the element of $C(v_k)$ that is obtained by repeatedly applying the Transition condition to δ . For shorthand, set $\alpha = C(v, e)$ and $\beta = C(v, f)$.

Case 1: $\tilde{\Phi}$ is of type $A_1 \times A_1$, so $\omega(\alpha^\vee, \beta) = 0$. Then

$$\delta_4 = \delta + [\omega(\alpha^\vee, \delta)]_+(\alpha) + [\omega(\beta^\vee, \delta)](\beta) + [\omega(\alpha^\vee, \delta)]_+(-\alpha) + [\omega(\beta^\vee, \delta)](-\beta) = \delta$$

as desired.

Case 2: $\tilde{\Phi}$ is not of type $A_1 \times A_1$, so the restriction of ω to \tilde{V} is nondegenerate. Then we can write $\delta = \nu + \kappa$ with $\kappa \in \tilde{V}$ and $\omega(\alpha^\vee, \nu) = \omega(\beta^\vee, \nu) = 0$. The piecewise linear transformations turning δ_j into δ_{j+1} all preserve the ν component and act solely on κ . Thus, it is enough to see that these piecewise linear transformations act on \tilde{V} with period $h + 2$.

Divide \tilde{V} into $h + 2$ cones, one spanned by each pair of adjacent almost positive roots. A case-by-case verification shows that the action on each of these cones is linear, and repeats after $h + 2$ steps. \square

We are now prepared to define a simply connected framework. As before, let (G, C, C^\vee) be a framework. Motivated by Lemma 4.14, define a 2-dimensional *CW*-complex Σ whose 1-skeleton is G and whose 2-faces have boundaries the rank two cycles. We define (G, C, C^\vee) to be **simply connected** if Σ is simply connected. Equivalently, (G, C, C^\vee) is simply connected if $\pi_1(G, v)$ is generated by paths of the form $\sigma\tau\sigma^{-1}$ where τ travels around a rank 2 cycle and σ is some path from the basepoint v to that rank two cycle. (This condition is plainly independent of the choice of basepoint v .)

Proposition 4.16. *If (G, C, C^\vee) is simply connected, then it is ample.*

Proof. Let \tilde{v}_{start} and \tilde{v}_{end} be two vertices of \hat{G} , lying above the same vertex v of G . Let $\tilde{\rho}$ be the path from \tilde{v}_{start} to \tilde{v}_{end} in \hat{G} , so $\tilde{\rho}$ projects down to a cycle ρ in G . So ρ can be written as the concatenation of paths of the form $\sigma\tau\sigma^{-1}$ as above. Let σ run from v to u . The cycle τ lifts to a path $\tilde{\tau}$ from some \tilde{u}_1 to some \tilde{u}_2 . Let σ lift to the path in \hat{G} from \tilde{v}_1 to \tilde{u}_1 , and let σ^{-1} lift to the path from \tilde{u}_2 to \tilde{v}_2 .

By Lemma 4.14, Seed takes the same value at \tilde{u}_1 and \tilde{u}_2 . Since mutation is involutive, traveling from \tilde{u}_2 to \tilde{v}_2 precisely undoes the effect on Seed of traveling from \tilde{v}_1 to \tilde{u}_1 . So $\text{Seed}(\tilde{v}_1) = \text{Seed}(\tilde{v}_2)$. Continuing in this manner, we deduce that $\text{Seed}(\tilde{v}_{\text{start}}) = \text{Seed}(\tilde{v}_{\text{end}})$, as desired. \square

Proposition 4.17. *Let (G, C, C^\vee) be a polyhedral framework, with corresponding fan \mathcal{F} , supported in V^* . We write $|\mathcal{F}|$ for the union of the cones in \mathcal{F} . Let S be a sphere around the origin in V^* . Let Ω be the open subset of $|\mathcal{F}| \cap S$ formed by deleting $F \cap S$ for any $F \in \mathcal{F}$ of codimension ≥ 3 . Then (G, C, C^\vee) is simply connected if and only if the topological space Ω is simply connected.*

Proof. Take an open covering of Ω , with one open set U_v for each vertex v of G , where U_v is a small thickening of $\text{Cone}(v) \cap \Omega$. Then we can compute $\pi_1(\Omega)$ using van Kampen's theorem for groupoids, and see that it is the same as $\pi_1(\Sigma)$. See, for example, [17]. \square

Remark 4.18. If $|\mathcal{F}| \cap S$ is a manifold with boundary, then the points that are deleted from this intersection in order to form Ω are sub-manifolds-with-boundary of codimension 3, so $\pi_1(|\mathcal{F}| \cap S) \cong \pi_1(\Omega)$. However, one could imagine that \mathcal{F} has

some codimension 3 face whose link is disconnected, in which case it is important to define Ω as above.

4.3. Descending frameworks. We now describe a condition that implies simple-connectivity, and many other good conditions, but requires no topological notions. We say a framework is *descending* if it satisfies the following three conditions.

Positive labels condition: If a vertex v of G has $\{\text{sgn}(\beta) : \beta \in C(v)\} = \{1\}$, then v is the base vertex v_b .

Half-edge condition: If e is a half-edge incident to v , then $\text{sgn}(C(v, e)) = 1$.

The Sign and Transition conditions let us give an orientation to each edge of G . If e is an edge incident to a vertex v , then we direct e towards v if $\text{sgn}(C(v, e)) = 1$ and away from v if $\text{sgn}(C(v, e)) = -1$.

Descending chain condition: There exists no infinite sequence $v_0 \rightarrow v_1 \rightarrow \dots$.

The Descending chain condition is unsatisfying, because it will be easy to see, by [11, Theorem 1.8] (and in particular the implication (iii) \implies (i) in that theorem), that a complete framework cannot satisfy the Descending chain condition unless B is of finite type. However, the notion of a descending framework will be critical to the construction of complete exact frameworks for exchange matrices B whose associated Cartan matrix is of finite or affine type, and the construction of (non-complete) exact frameworks in general. The key point is the following theorem.

Theorem 4.19. *A descending framework is exact, polyhedral, and well-connected.*

Recall that the polyhedral property implies injectivity. Thus we need only prove that a descending framework is ample, polyhedral, and well-connected. We prove this as three separate propositions.

Proposition 4.20. *A descending framework is polyhedral.*

We first need a lemma:

Lemma 4.21. *Suppose (G, C, C^\vee) is a directed framework. For each vertex v of G , there are a finite, nonzero number of directed paths from v to v_b .*

Proof. The Half-Edge condition and the Positive labels condition imply that every vertex $v \neq v_b$ has an edge directed from v to another vertex v' . This observation and the Descending chain condition imply that, for every vertex v , there is a finite directed path from v to v_b . On the other hand, suppose there are infinitely many directed paths from v to v_b . The vertex v has out-degree at most n . Thus some vertex v' with $v \rightarrow v'$ has infinitely many directed paths from v' to v_b . For the same reason, there is some vertex v'' with $v' \rightarrow v''$ such that there are infinitely many directed paths from v'' to v_b . Continuing in this way, we obtain a contradiction to the Descending chain condition. \square

In light of Lemma 4.21, we can define the *length* $\ell(v)$ of v to be the length of a longest directed path from v to v_b .

Proof of Proposition 4.20. We first check that, if $\text{Cone}(u) = \text{Cone}(v)$, then $u = v$. Our proof is by induction on $\ell(u) + \ell(v)$; the base case is trivial. If $\ell(u) + \ell(v) > 0$

then one of u and v , say without loss of generality u , must not be the base vertex v_b . Take a path $u \rightarrow u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow v_b$. Let e be the edge $u \rightarrow u_1$. Since $\text{Cone}(u) = \text{Cone}(v)$, there is an edge f incident to v with $C(v, f) = C(u, e)$. By the Half-edge condition, there is a vertex v_1 at the other end of f . Then, $\text{Cone}(u_1) = \text{Cone}(v_1)$ as they are computed from $\text{Cone}(u)$ and $\text{Cone}(v)$ by the same recursion. By induction, $u_1 = v_1$. Then e and f are two edges incident to u_1 with $C(u_1, e) = C(u_1, f)$, so $e = f$ and $u = v$, as desired.

So the vertices of G are in bijection with the set of cones $\{\text{Cone}(v) : v \in G\}$. We now must check that these cones are the maximal faces of a fan. In light of Lemma 3.6, we need simply check that, for any two distinct vertices u and v of G , the cones $\text{Cone}(u)$ and $\text{Cone}(v)$ meet nicely.

Suppose for the sake of contradiction that there exist distinct vertices u and v such that $\text{Cone}(u)$ and $\text{Cone}(v)$ do not meet nicely and choose u and v so as to minimize $\ell(u) + \ell(v)$. We will say that two cones *meet badly* if they do not meet nicely.

We consider two cases. Throughout the argument, x will be a point in the interior of $\text{Cone}(v_b)$.

Case 1: The cones $\text{Cone}(u)$ and $\text{Cone}(v)$ intersect in dimension n . Since u and v are distinct, at least one of them has positive length, and therefore by the Positive labels condition, there is at least one element of $C(u) \cup C(v)$ whose sign is -1 . Let p be a point in the intersection of the interior of $\text{Cone}(u)$ with the interior of $\text{Cone}(v)$. Consider points of the form $p + \varepsilon x$ for $\varepsilon > 0$. If $\beta \in C(u) \cup C(v)$, then the set $\{p + \varepsilon x : \varepsilon > 0\}$ intersects the hyperplane $\beta^\perp \subset V^*$ if and only if $\text{sgn}(\beta) = -1$. By choosing p generically, we can assume that $\{p + \varepsilon x : \varepsilon > 0\}$ intersects each of the hyperplanes β^\perp with $\beta \in C(u) \cup C(v)$ and $\text{sgn}(\beta) = -1$ at a different point (except if two of the hyperplanes coincide). Let ε_0 be the smallest positive ε such that $p + \varepsilon x$ is contained in a hyperplane β^\perp with $\beta \in C(u) \cup C(v)$. Then $p + \varepsilon_0 x$ is contained in the relative interior of a facet of $\text{Cone}(u)$ or a facet of $\text{Cone}(v)$ or both.

We first consider the case where $p + \varepsilon_0 x$ is contained both in the relative interior of a facet F of $\text{Cone}(u)$ and in the relative interior of a facet G of $\text{Cone}(v)$. By the Half-edge condition, the edge in $I(u)$ labeled β is directed from u to a vertex u' , and the edge in $I(v)$ labeled β is directed from v to a vertex v' . By Corollary 2.8, $\text{Cone}(u)$ and $\text{Cone}(u')$ share the facet F and $\text{Cone}(v)$ and $\text{Cone}(v')$ share the facet G . The facets F and G are defined by the same hyperplane. If $u' = v'$, then u' is incident to two edges with the label $-\beta$, contradicting Proposition 2.3. Otherwise, for small enough $\varepsilon > \varepsilon_0$, the point $p + \varepsilon x$ is in the intersection of the interior of $\text{Cone}(u')$ with the interior of $\text{Cone}(v')$. This contradicts our choice of u and v to minimize $\ell(u) + \ell(v)$.

Now we can assume, without loss of generality, that $p + \varepsilon_0 x$ is in the interior of $\text{Cone}(v)$ and in the relative interior of a facet F of $\text{Cone}(u)$. The Half-edge condition says that the edge in $I(u)$ labeled β is directed from u to a vertex u' , and Corollary 2.8 says that $\text{Cone}(u)$ and $\text{Cone}(u')$ share the facet F . Thus for small enough $\varepsilon > \varepsilon_0$, the point $p + \varepsilon x$ is in the intersection of the interior of $\text{Cone}(u')$ with the interior of $\text{Cone}(v)$. Again, this contradicts our choice of u and v .

Case 2: The cones $\text{Cone}(u)$ and $\text{Cone}(v)$ intersect in dimension less than n . Let F_1, \dots, F_k be the set of facets of $\text{Cone}(u)$ containing $\text{Cone}(u) \cap \text{Cone}(v)$, and let G_1, \dots, G_l be the set of facets of $\text{Cone}(v)$ containing $\text{Cone}(u) \cap \text{Cone}(v)$. Each of

the facets in $\{F_1, \dots, F_k, G_1, \dots, G_l\}$ is defined by a vector $\beta \in C(v) \cap C(v)$. We claim that at least one of these vectors β has $\text{sgn}(\beta) = -1$.

Suppose for the sake of contradiction that the claim fails. Let p be a point in the relative interior of $\text{Cone}(u) \cap \text{Cone}(v)$. Since each F_i is defined by a vector β with $\text{sgn}(\beta) = 1$, the vector $p + \varepsilon x$ is in the interior of $\text{Cone}(u)$ for small enough positive ε . For the same reason, $p + \varepsilon x$ is in the interior of $\text{Cone}(v)$ for small enough positive ε . This shows that the interior of $\text{Cone}(u)$ intersects the interior of $\text{Cone}(v)$, contradicting the hypothesis of Case 2, and thus proving the claim.

Without loss of generality, let F be a facet of $\text{Cone}(u)$ that contains $\text{Cone}(u) \cap \text{Cone}(v)$ and such that F is defined by $\beta \in C(u)$ with $\text{sgn}(\beta) = -1$. Then $F \cap \text{Cone}(v) = \text{Cone}(u) \cap \text{Cone}(v)$, and this intersection is either not a face of F or not a face of $\text{Cone}(v)$ or both. In other words, F and $\text{Cone}(v)$ meet badly. The Half-edge condition says that the edge in $I(u)$ labeled β is directed from u to a vertex u' . By Corollary 2.8, $\text{Cone}(u)$ and $\text{Cone}(u')$ share the facet F . Since F and $\text{Cone}(v)$ meet badly, $\text{Cone}(u')$ and $\text{Cone}(v)$ meet badly. This contradicts our choice of u and v to minimize $\ell(u) + \ell(v)$, thus completing the proof. \square

Proposition 4.22. *A descending framework is well-connected.*

Proof. We first show that there exists a path $u = u_0, u_1, \dots, u_l$ such that F is a face of $\text{Cone}(u_i)$ for all i from 0 to l . More specifically, we show that $F = \text{Cone}(u_l) \cap \beta_1^\perp \cap \dots \cap \beta_p^\perp$ such that β_1, \dots, β_p are labels in $C(u_l)$, each with sign +1. If the singleton path u does not have this property, then there is a label $\beta \in C(u)$ with $\text{sgn}(\beta) = -1$ such that $F \subset \beta^\perp$. The Half-edge condition implies that β is $C(u, e)$ where e is an edge connecting u to a vertex u_1 . By Corollary 2.8, F is a face of $\text{Cone}(u_1)$. If the path $u = u_0, u_1$ does not have the desired property, then we construct u_2 , etc. The Descending Chain condition implies that eventually we will construct a path with the desired property. In particular, if y is any point in the relative interior of F and x is a point in the interior of $\text{Cone}(v_b)$, then for small enough ε the point $y + \varepsilon x$ is in the interior of $\text{Cone}(u_l)$.

We can now perform the same construction to obtain a path $v = v_0, v_1, \dots, v_m$ such that F is a face of $\text{Cone}(v_i)$ for all i from 0 to m , and $F = \text{Cone}(v_m) \cap \gamma_1^\perp \cap \dots \cap \gamma_q^\perp$ such that $\gamma_1, \dots, \gamma_q$ are labels in $C(v_m)$, each with sign +1. But then, for small enough ε the point $y + \varepsilon x$ is in the interior of $\text{Cone}(v_m)$. By Proposition 4.20, we conclude that $u_l = v_m$. Now, concatenating the two paths, we obtain the desired path between u and v . \square

The following proposition combines with Proposition 4.16 to show that a descending framework is ample.

Proposition 4.23. *A descending framework is simply connected.*

The outline of the proof of Proposition 4.23 is a simple inductive argument once we have the following lemma.

Lemma 4.24. *Suppose (G, C, C^\vee) is a descending framework and suppose u, v and w are vertices of G with $v \rightarrow u$ and $v \rightarrow w$. Then there exists a rank-two cycle in G containing u, v and w that has v as its unique source.*

Proof. In the notation of Section 4.2, let e and f be the edges $v \rightarrow u$ and $v \rightarrow w$. Form the sequences v_k, e_k and γ_k as in that section; although we don't know yet that those sequences are bi-infinite. Let b and c be as before.

First, suppose for the sake of contradiction that $\begin{pmatrix} 2 & -|b| \\ -|c| & 2 \end{pmatrix}$ is of infinite type. Then all of the γ_k for $k > 0$ are negative roots and, by the Half-edge condition, our sequence cannot terminate in the direction of positive k . But then $v_0 \rightarrow v_1 \rightarrow v_2 \rightarrow \cdots$ is an infinite descending chain, a contradiction.

So $\begin{pmatrix} 2 & -|b| \\ -|c| & 2 \end{pmatrix}$ is of finite type, let its Coxeter number be h . Switching u and w if necessary, we may assume that $b \geq 0$ and $c \leq 0$. Then the computations of Section 4.2 show that we have $v_{-2} \leftarrow v_{-1} \leftarrow v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_h$, and all of these edges exist by the Half-edge condition. Moreover, from Lemma 4.15, we have $C(v_{-2}) = C(v_h)$. So $\text{Cone}(v_{-2}) = \text{Cone}(v_h)$ and, as (G, C, C^\vee) is polyhedral, we have $v_{-2} = v_h$. So $v_{-2} \leftarrow v_{-1} \leftarrow v_0 \rightarrow v_1 \rightarrow \cdots \rightarrow v_h = v_{-2}$ is the desired cycle. \square

Proof of Proposition 4.23. We will show that any path v_0, \dots, v_k from v_b to itself can be contracted in Σ . Let m be the maximum length of a vertex on the path. We argue by induction on m and, for fixed m , by induction on the number r of times the length m is achieved on the path. If $m = 0$ then the path is the singleton at v_b . Otherwise, let v_i be a vertex in the path with $\ell(v) = m > 0$. Since m is the maximum length on the path, the neighbors of v_i in the path must have lower length, so the edges to these neighbors are directed $v_i \rightarrow v_{i-1}$ and $v_i \rightarrow v_{i+1}$. If $v_{i-1} = v_{i+1}$ then the path backtracks at v_i and we may delete this backtracking and proceed by induction.

If $v_{i-1} \neq v_{i+1}$, Lemma 4.24 says that there exists a rank-two cycle in G , having v_i as its unique source. In particular, every vertex on that cycle has length strictly less than $\ell(v)$. Now replace the segment (v_{i-1}, v_i, v_{i+1}) by the other h edges in that cycle. By construction, the new path is related to the old path by moving across a face of Σ .

If $r > 1$, then the new path has maximum length m , realized $r - 1$ times. If $r = 1$, then the new path has maximum length $m - 1$. In any case, by induction, the new path can be contracted in Σ . \square

We conclude our discussion of descending frameworks with one more property.

Proposition 4.25. *Let (G, C, C^\vee) be a descending framework. For each vertex v of G , the \mathbf{g} -vectors of the cluster variables in $\text{Seed}(v)$ form a sign-coherent set.*

Proof. In light of Theorem 3.24(3), the assertion that the \mathbf{g} -vectors for v are sign-coherent is equivalent to the following assertion: For each $i \in I$, the interior of $\text{Cone}(v)$ is disjoint from the hyperplane α_i^\perp .

Suppose the assertion fails for some i , and choose v to minimize $\ell(v)$ among counterexamples. We know that $v \neq v_b$ because $\text{Cone}(v_b) = D$. Let y be a point in the intersection of α_i^\perp and the interior of $\text{Cone}(v)$. Since $y \in \alpha_i^\perp$, expanding y in the basis of fundamental weights, the coefficient of ρ_i is 0. Let x be $\sum_{j \neq i} \rho_j$, the sum of the fundamental weights besides ρ_i . Then for a large enough scalar a , the vector $y + ax$ is in D . Proposition 4.20 implies that the interior of $\text{Cone}(v)$ is disjoint from $D = \text{Cone}(v_b)$. Thus the line segment connecting y to $y + ax$ exits the interior of $\text{Cone}(v)$ through the relative interior of some face F of $\text{Cone}(v)$. Let β_1, \dots, β_k be the distinct vectors in $C(v)$ such that $F = \text{Cone}(v) \cap (\beta_1^\perp \cap \cdots \cap \beta_k^\perp)$. Since ax is a positive combination of fundamental weights and the line segment from y to $y + ax$ crosses each hyperplane β_j^\perp , we conclude that $\text{sgn}(\beta_j) = 1$ for each j .

By the argument in the proof of Proposition 4.22, we know that there exists a path $v = v_0, v_1, \dots, v_l$ such that F is the face $\text{Cone}(v_l) \cap \gamma_1^\perp \cap \dots \cap \gamma_p^\perp$ of $\text{Cone}(v_l)$, where $\gamma_1, \dots, \gamma_p$ are labels in $C(v_l)$, each with sign $+1$. The argument furthermore establishes that the path is directed as $v = v_0 \rightarrow v_1 \rightarrow \dots \rightarrow v_l$. Thus $\ell(v_l) < \ell(v)$.

We now see that upon exiting the interior of $\text{Cone}(v)$ through the relative interior of F , the line segment from y to $y + ax$ enters the interior of $\text{Cone}(v_l)$. Since this line segment is contained in α_i^\perp and $\ell(v_l) < \ell(v)$, we have obtained a contradiction to our choice of v . \square

5. CAMBRIAN FRAMEWORKS

In this section, we apply Cambrian combinatorics to construct a descending reflection framework for a given acyclic exchange matrix B . We have seen that B determines a Cartan matrix A and thus a Coxeter group W . The matrix B also determines a Coxeter element of W , in a way that we review below. The graph G in the framework is the Hasse diagram of the **Cambrian semilattice** associated to the orientation. Thus the vertices of G are the **sortable elements**. The labels are certain roots that can be read off combinatorially from sorting words for the sortable elements. The framework is not complete unless B is of finite Cartan type, but in a future paper we will extend the construction to produce complete frameworks in the case where A is of affine type.

More details on the combinatorial and polyhedral constructions in this section can be found in [25].

5.1. Sortable elements and Cambrian lattices. Let W be the Coxeter group determined by B as explained in Section 2.2. Recall that $S = \{s_i : i \in I\}$ is the set of simple reflections in W . It will be convenient, in what follows, to sometimes suppress the indexing set I and let S serve as an indexing set. Thus, for example, we may write α_s for α_i when $s = s_i$, and so forth.

When B is acyclic, it encodes a **Coxeter element** c of W . A Coxeter element is an element that arises by multiplying the generators S , in any order, with each generator appearing exactly once. Since B is acyclic, we can take I to be $\{1, \dots, n\}$ such that $B_{ij} > 0$ implies $i < j$ and let c be the Coxeter element $s_1 \cdots s_n$.

An expression for $w \in W$ as a word in the generators S is called **reduced** if it is of minimal length among words for w . This minimal length is called the **length** of w and written $\ell(w)$. The **(right) weak order** on W is the transitive closure of the relations $w < ws$ for all $w \in W$ and $s \in S$ such that $\ell(w) < \ell(ws)$. In this paper, inequalities between elements of W always mean this relation.

The weak order is a meet-semilattice, and furthermore, given any subset $U \subseteq W$, if U has an upper bound in W , then it has a join $\bigvee U$. The weak order is also characterized in terms of inversion sets. An **inversion** of $w \in W$ is a reflection t such that $\ell(tw) < \ell(w)$. Write $\text{inv}(w)$ for the set of inversions of w . We have $u \leq w$ if and only if $\text{inv}(u) \subseteq \text{inv}(w)$.

Let c^∞ be the half-infinite word $s_1 \cdots s_n s_1 \cdots s_n s_1 \cdots s_n \dots$ formed by infinitely repeating the word $s_1 \cdots s_n$. Given $w \in W$, the **c -sorting word** for w is the word obtained by choosing, among all subsequences of c^∞ that form reduced words for w , the subsequence that is lexicographically leftmost in c^∞ . Every element of w has a unique c -sorting word. This word is equivalent to a sequence of subsets of S : Reading c^∞ from left to right, in each repetition of $s_1 \cdots s_n$, we take the set of

letters that appear in the c -sorting word for w . If this sequence of subsets is weakly decreasing in containment order, then we call w a **c -sortable element**.

A generator $s \in S$ is **initial** in c if there is a reduced word for c having s as its first letter. Similarly, s is **final** in c if it is the last letter of some reduced word for c . In either case, the element scs is another Coxeter element.

Given a subset $J \subseteq S$, the **standard parabolic subgroup** W_J is the subgroup of W generated by J . This subgroup forms an order ideal in the weak order on W . The **restriction** of c to W_J is the Coxeter element of W_J obtained by deleting the letters in $S \setminus J$ from any reduced word for c . If $w \in W$ then there exists a unique element, denoted w_J , such that $\text{inv}(w_J) = \text{inv}(w) \cap W_J$. We will be most interested in the case where $J = S \setminus \{s\}$, and we define the special notation $\langle s \rangle$ to stand for $S \setminus \{s\}$.

The next two lemmas are [22, Lemmas 2.4, 2.5]. Since the identity element is c -sortable for any c , the lemmas are a recursive characterization of c -sortability, by induction on the length $\ell(w)$ and on the rank of W (the cardinality of S).

Lemma 5.1. *Let s be initial in c and suppose $w \not\geq s$. Then w is c -sortable if and only if it is an sc -sortable element of $W_{\langle s \rangle}$.*

Lemma 5.2. *Let s be initial in c and suppose $w \geq s$. Then w is c -sortable if and only if sw is scs -sortable.*

The following is [25, Proposition 3.13].

Proposition 5.3. *If v is c -sortable and $J \subseteq S$, then v_J is c' -sortable, where c' is the restriction of c to W_J .*

Let v be a c -sortable element of W and let $a_1 \cdots a_k$ be its c -sorting word. For $r \in S$, there is a leftmost instance of r in c^∞ that is not in the subword of c^∞ corresponding to $a_1 \cdots a_k$. Let $a_1 \cdots a_j$ be the initial segment of $a_1 \cdots a_k$ consisting of those letters that occur in c^∞ before the omission of r . Say $a_1 \cdots a_k$ **skips** r after $a_1 \cdots a_j$. If $a_1 \cdots a_j r$ is a reduced word, then this is an **unforced skip**. Otherwise it is a **forced skip**. Define $C_c^r(v)$ to be the root $a_1 \cdots a_j \cdot \alpha_r$. This is a positive root if and only if the skip is unforced. Although this definition refers to the position of $a_1 \cdots a_k$ in c^∞ , the root $C_c^r(v)$ is read off easily from $a_1 \cdots a_k$ itself. Write $C_c(v)$ for $\{C_c^r(v) : r \in S\}$.

This definition of $C_c(v)$ is shown in [25, Proposition 5.1] to be equivalent to the following recursive definition: For s initial in c ,

$$C_c(v) = \begin{cases} C_{sc}(v) \cup \{\alpha_s\} & \text{if } v \not\geq s \\ sC_{scs}(sv) & \text{if } v \geq s \end{cases}$$

The sets $C_{sc}(v)$ and $C_{scs}(sv)$ are defined by induction on the rank of W or on the length of v .

A **cover reflection** of $w \in W$ is an inversion t of w such that $tw = ws$ for some $s \in S$. The name “cover reflection” refers to the fact that w covers tw in the weak order. Indeed, the cover reflections of w are the elements $ws w^{-1}$ such that $s \in S$ and $ws < w$. The set of cover reflections of w is written $\text{cov}(w)$. If t is a cover reflection of w then $\text{inv}(tw) = \text{inv}(w) \setminus \{t\}$.

The following is [25, Proposition 5.2].

Proposition 5.4. *Let v be a c -sortable element. The set of negative roots in $C_c(v)$ is $\{-\beta_t : t \in \text{cov}(v)\}$.*

The *c-Cambrian semilattice* Camb_c is the subposet of the weak order on W induced by the c -sortable elements. It is a sub-meet-semilattice of the weak order on W by [25, Theorem 7.1]. We will also use the symbol Camb_c to denote the undirected Hasse diagram of Camb_c .

5.2. The Cambrian framework. In this section, we show that (Camb_c, C_c) is, in essence, a descending reflection framework. But there is a little more work to do before we can make a precise statement. Specifically, we need to add some half-edges to the Hasse diagram of Camb_c to get an n -regular quasi-graph. Also, as it stands, the labels C_c are not assigned to edges incident to a vertex v , but rather are indexed by S . To fill in these pieces of the Cambrian framework, we will need more background on sortable elements and Cambrian lattices.

Suppose w is any element of W . Among the c -sortable elements that are $\leq w$, there is a maximal one ([25, Corollary 6.2]); we denote this maximal element by $\pi_{\downarrow}^c(w)$. One can also define π_{\downarrow}^c recursively; see [25, Section 6]. The following is [25, Theorem 7.3].

Theorem 5.5. *For U any subset of W , if U is nonempty then $\bigwedge \pi_{\downarrow}^c(U) = \pi_{\downarrow}^c(\bigwedge U)$ and if U has an upper bound then $\bigvee \pi_{\downarrow}^c(U) = \pi_{\downarrow}^c(\bigvee U)$.*

The following is [25, Theorem 6.1], or it can be obtained as an immediate corollary of Theorem 5.5.

Theorem 5.6. *π_{\downarrow}^c is order preserving.*

We now describe the cover relations in Camb_c .

Lemma 5.7. *Let v be a c -sortable element. Then $v' < v$ in the c -Cambrian semilattice if and only if $v' = \pi_{\downarrow}^c(tv)$ for some $t \in \text{cov}(v)$. Furthermore if t_1 and t_2 are distinct cover reflections of v , then $\pi_{\downarrow}^c(t_1v) \neq \pi_{\downarrow}^c(t_2v)$.*

Lemma 5.7 follows from a general result (see e.g. [20, Proposition 2.2]) on lattice congruences on a finite lattice, because Theorem 5.5 implies that the fibers of π_{\downarrow}^c are a lattice congruence on the interval $[1, v]$. We provide a direct proof as well.

Proof. We first verify the second assertion by proving the stronger statement that $\pi_{\downarrow}^c(t_1v)$ and $\pi_{\downarrow}^c(t_2v)$ are incomparable. Let t_1 and t_2 be distinct cover reflections of v . Then the join of t_1v and t_2v exists and equals v , so Theorem 5.5 says that

$$v = \pi_{\downarrow}^c(v) = \pi_{\downarrow}^c(t_1v \vee t_2v) = \pi_{\downarrow}^c(t_1v) \vee \pi_{\downarrow}^c(t_2v).$$

Since $t_1v < v$ and $t_2v < v$, Theorem 5.6 says that $\pi_{\downarrow}^c(t_1v) < v$ and $\pi_{\downarrow}^c(t_2v) < v$. The elements $\pi_{\downarrow}^c(t_1v)$ and $\pi_{\downarrow}^c(t_2v)$ must in particular be incomparable to join to v .

Suppose $v' < v$ in Camb_c . Then $v' \leq v$ in the weak order, so there exists an element $w \in W$ with $v' \leq w < v$ in the weak order, and necessarily, $w = tv$ for some $t \in \text{cov}(v)$. The element $\pi_{\downarrow}^c(tv)$ has $v' \leq \pi_{\downarrow}^c(tv) < v$ in Camb_c . Since $v' < v$ in Camb_c , we conclude that $v' = \pi_{\downarrow}^c(tv)$.

Finally, consider any $t \in \text{cov}(v)$. By Theorem 5.6, $\pi_{\downarrow}^c(tv) < v$. Thus there exists a c -sortable element v' with $\pi_{\downarrow}^c(tv) \leq v' < v$ in Camb_c . By the previous paragraph, $v' = \pi_{\downarrow}^c(t'v)$ for some $t' \in \text{cov}(v)$. But we showed above that $\pi_{\downarrow}^c(tv)$ and $\pi_{\downarrow}^c(t'v)$ are incomparable if $t' \neq t$. We conclude that $t' = t$, so that $\pi_{\downarrow}^c(tv) = v'$. \square

For each c -sortable element v , define a cone

$$\text{Cone}_c(v) = \bigcap_{\beta \in C_c(v)} \{x \in V^* : \langle x, \beta \rangle \geq 0\}.$$

Once we have completed the construction of a Cambrian framework, this $\text{Cone}_c(v)$ will coincide with the cone $\text{Cone}(v)$ defined in Section 2. Recall the cone $D = \bigcap_{s \in S} \{x \in V^* : \langle x, \alpha_s \rangle \geq 0\}$. The cones wD , for $w \in W$, are distinct, and form a fan whose support is called the **Tits cone**. The following is [25, Theorem 6.3].

Theorem 5.8. *Let v be c -sortable. Then $\pi_\downarrow^c(w) = v$ if and only if $wD \subseteq \text{Cone}_c(v)$.*

The following lemma is the last step needed to let us make a precise statement about Cambrian frameworks.

Lemma 5.9. *If $v' < v$ in the c -Cambrian semilattice, then there exists a unique root β such that $\beta \in C_c(v')$ and $-\beta \in C_c(v)$. The root β is the positive root β_t associated to the cover reflection t of v such that $v' = \pi_\downarrow^c(tv)$.*

Proof. By Lemma 5.7, $v = \pi_\downarrow^c(tv)$ for some $t \in \text{cov}(v)$. By Theorem 5.8, $vD \subseteq \text{Cone}_c(v)$ and $tvD \subseteq \text{Cone}_c(v')$, and furthermore, $vD \not\subseteq \text{Cone}_c(v')$ and $tvD \not\subseteq \text{Cone}_c(v)$. We know that vD and tvD share a codimension-1 facet contained in $(\beta_t)^\perp$, so the cones $\text{Cone}_c(v)$ and $\text{Cone}_c(v')$ each have a facet contained in $(\beta_t)^\perp$. So one of $\{\beta_t, -\beta_t\}$ is in $C_c(v')$ and the other is in $C_c(v)$. Furthermore, $\text{Cone}_c(v)$ is separated from D by $(\beta_t)^\perp$, so $\beta_t \in C_c(v')$ and $-\beta_t \in C_c(v)$.

Since vD and tvD are not separated by any hyperplane besides $(\beta_t)^\perp$, there is no other hyperplane separating $\text{Cone}_c(v)$ from $\text{Cone}_c(v')$ and thus β_t is the unique root β , as desired. \square

We describe how to use Lemma 5.9 to associate a root to each incident pair in Camb_c . Suppose $v' < v$ in Camb_c with $v' = \pi_\downarrow^c(tv)$ and write e for the edge (v, v') . We label the incident pair (v', e) by the root β_t and label (v, e) by the root $-\beta_t$. This assigns some of the roots in $C_c(v)$ to edges incident to v , (and does not assign the two roots to the same edge). In particular that the degree of v in Camb_c is at most n . If the degree is less than n , then we affix half-edges to v to adjust the degree of v to be n . These half-edges are labeled with the remaining roots from $C_v(v)$. We again re-use the symbol Camb_c for the quasi-graph thus obtained. We also re-use the symbol C_c to denote the labeling of incident pairs of Camb_c by roots. We can now make a precise statement about Cambrian frameworks.

Theorem 5.10. *The pair (Camb_c, C_c) is a descending reflection framework for the exchange matrix B .*

Combining Theorem 5.10 with Theorem 4.19, we obtain the following result, which was conjectured in [25, Section 9.1]. Let \mathcal{F}_c be the collection consisting of all of the cones $\text{Cone}_c(v)$ together with their faces, where v ranges over all c -sortable elements. We call \mathcal{F}_c the **c -Cambrian fan**.

Corollary 5.11. *The collection \mathcal{F}_c is a fan, and distinct vertices of Camb_c label distinct maximal cones of \mathcal{F}_c .*

Combining Theorem 5.10 with Proposition 4.25, we obtain the following result.

Corollary 5.12. *Given any vertex v in the Cambrian framework (Camb_c, C_c) , the \mathbf{g} -vectors of the cluster variables in $\text{Seed}(v)$ form a sign-coherent set.*

When B is of finite Cartan type, W is of finite type. In this case Camb_c is the **Cambrian lattice**, rather than *semilattice*, and \mathcal{F}_c coincides, via Theorem 5.8, to the fan defined by the **Cambrian congruence**. See [21, 23] for details on the Cambrian congruence, and see [20] for details on the construction of a fan from a lattice congruence on the weak order.

Also, when B is of finite Cartan type, the Hasse diagram of the Cambrian lattice is an n -regular graph [24, Corollary 8.1], so the framework (Camb_c, C_c) is complete. Thus Theorem 5.10 combines with Theorem 4.1, Corollaries 4.4 and 4.6 and Proposition 4.25 to give the following result.

Corollary 5.13. *If B is of finite Cartan type, then Conjectures 3.7, 3.8, 3.9, 3.10, 3.11, 3.12, 3.13, 3.14, 3.15, 3.16, 3.17, and 3.18 all hold for B .*

In Section 5.3, we prove Conjectures 3.19, 3.20, 3.21 and 3.22 as well for B of finite Cartan type. Conjectures 3.9, 3.10, and 3.19 are new for B of finite Cartan type, as far as the authors know. The other conjectures appearing in Corollary 5.13 are already known when B is of finite Cartan type, as explained at the end of Section 3.3.

Remark 5.14. This proof of Conjecture 3.15 for B of finite Cartan type is not entirely new. The combination of [20, Theorem 1.1] with [23, Theorem 1.1] amounts to a proof that the c -Cambrian fan is indeed a fan for B of finite Cartan type. It was conjectured in [24, Section 10] that the c -Cambrian fan coincides with the collection of \mathbf{g} -vector cones, and proven for a special choice of c . The conjecture was proved for all c by Yang and Zelevinsky in [27].

When B is of infinite Cartan type, the framework (Camb_c, C_c) is not complete. Indeed, as mentioned in Section 4.3, a descending framework cannot be complete unless B is of finite type. But there is a deeper reason for the incompleteness. Theorem 5.8 implies in particular that each cone $\text{Cone}_c(v)$ intersects the Tits cone. Typically, there are \mathbf{g} -vector cones that don't intersect the Tits cone.

We now proceed to prove Theorem 5.10. To begin with, the pair (Camb_c, C_c) satisfies the Base condition because, when v is the identity element, $C_c(v) = \{\alpha_i : i \in I\}$. The pair satisfies the Root condition by construction.

The form ω agrees with the form ω_c defined in [25, Section 3]. Thus, the following proposition establishes the Reflection condition for (Camb_c, C_c) .

Proposition 5.15. *Suppose $v' \leq v$ in the c -Cambrian semilattice and let e be the edge connecting them. Let t be the reflection in the statement of Lemma 5.9, so that $C_c(v, e) = -\beta_t$ and $C_c(v', e) = \beta_t$. Let γ be any other root in $C_c(v)$. Then*

- (1) *If $\omega_c(\beta_t^\vee, \gamma) \geq 0$ then $t\gamma \in C_c(v')$.*
- (2) *If $\omega_c(\beta_t^\vee, \gamma) < 0$ then $\gamma \in C_c(v')$.*

The three lemmas below are [25, Lemmas 3.7–3.9]. Proposition 5.19 is a combination of parts of [25, Proposition 5.1] and [25, Proposition 5.4]. The subspace V_J of V is the real linear span of the simple roots $\{\alpha_s : s \in J\}$.

Lemma 5.16. *Let $J \subseteq S$ and let c' be the restriction of c to W_J . Then ω_c restricted to V_J is $\omega_{c'}$.*

Lemma 5.17. *If s is initial or final in c , then $\omega_c(\beta, \gamma) = \omega_{scs}(s\beta, s\gamma)$ for all roots β and γ .*

Lemma 5.18. *If s be initial in c and t is a reflection in W , then $\omega_c(\alpha_s, \beta_t) \geq 0$, with equality only if s and t commute.*

Proposition 5.19. *Let s be initial in c and let v be a c -sortable element of W such that s is a cover reflection of v . Then $\text{cov}(v) = \{s\} \cup \text{cov}(v_{\langle s \rangle})$, and the set of positive roots in $C_c(v)$ is obtained by applying the reflection s to each positive root in $C_{sc}(v_{\langle s \rangle})$.*

Using the above lemmas and proposition, we now prove Proposition 5.15.

Proof of Proposition 5.15. Our proof is by induction on the rank of W and the length of v . Let s be initial in c .

Case 1: $v' \geq s$. Then $v \geq s$. Since $s \in \text{inv}(v')$, and $t \notin \text{inv}(v')$, we have $s \neq t$. The recursive definition of C_c says that $s(-\beta_t)$ and $s\gamma$ are in $C_{scs}(sv)$ and that $s\beta_t \in C_{scs}(sv')$. As $s \neq t$, we have $s(-\beta_t) = -\beta_{sts}$ and $s\beta_t = \beta_{sts}$. By induction on $\ell(v)$, we conclude that

- (1) If $\omega_{scs}(s\beta_t^\vee, s\gamma) \geq 0$ then $(sts)s\gamma \in C_{scs}(sv')$.
- (2) If $\omega_{scs}(s\beta_t^\vee, s\gamma) < 0$ then $s\gamma \in C_{scs}(sv')$.

Lemma 5.17 says that $\omega_{scs}(s\beta_t^\vee, s\gamma) = \omega_c(\beta_t^\vee, \gamma)$, and we apply the recursive definition of C_c to obtain the desired conclusion in this case.

Case 2: $v \not\geq s$. Then $v' \not\geq s$. Since $s \notin \text{inv}(v)$, and $t \in \text{inv}(v)$, we have $s \neq t$. Both v and v' are sc -sortable elements of $W_{\langle s \rangle}$ by Lemma 5.1. Also, $C_c(v) = C_{sc}(v) \cup \{\alpha_s\}$ and $C_c(v') = C_{sc}(v') \cup \{\alpha_s\}$. Since $t \neq s$, we have $t \in W_J$.

If $\gamma = \alpha_s$, then Lemma 5.18 and the antisymmetry of ω_c say that $\omega_c(\beta_t^\vee, \gamma) \leq 0$, with equality if and only if s and t commute. If the inequality is strict, then the conclusion of the proposition holds because $\gamma = \alpha_s \in C_c(v')$. If equality holds, then s and t commute, so $\gamma = t\gamma = t\alpha_s = \alpha_s \in C_c(v')$.

If $\gamma \neq \alpha_s$, then γ is associated to a reflection in $W_{\langle s \rangle}$, so by induction on the rank of W , we see that

- (1) If $\omega_{sc}(\beta_t^\vee, \gamma) \geq 0$ then $t\gamma \in C_{sc}(v')$.
- (2) If $\omega_{sc}(\beta_t^\vee, \gamma) < 0$ then $\gamma \in C_{sc}(v')$.

By Lemma 5.16, we obtain the conclusion of the proposition.

Case 3: $v \geq s$ and $v' \not\geq s$. Then the hyperplane separating vD from tvD is $(\alpha_s)^\perp$, so $t = s$. We claim that $v' = v_{\langle s \rangle}$. Indeed, $\text{inv}(sv) = \text{inv}(v) \setminus \{s\}$, so $\text{inv}((sv)_{\langle s \rangle}) = \text{inv}(sv) \cap W_{\langle s \rangle} = \text{inv}(v) \cap W_{\langle s \rangle} = \text{inv}(v_{\langle s \rangle})$. Thus $(sv)_{\langle s \rangle} = v_{\langle s \rangle}$. This is the maximal element of $W_{\langle s \rangle}$ below sv , and it is also sc -sortable by Proposition 5.3. By Lemma 5.1, every c -sortable element below sv is in $W_{\langle s \rangle}$, so $v_{\langle s \rangle}$ is the maximal such, i.e. $v' = \pi_\downarrow^c(sv) = v_{\langle s \rangle}$.

Now $\beta_t = \alpha_s$, so Lemma 5.18 implies that $\omega_c(\beta_t^\vee, \gamma)$ agrees weakly in sign with $\text{sgn}(\gamma)$. In this case, we obtain the desired conclusion by combining Proposition 5.4 with Proposition 5.19. \square

We now complete the proof that (Camb_c, C_c) is a reflection framework by verifying the Euler conditions (E1), (E2), and (E3). The form E agrees with the form E_c defined in [25, Section 3]. Condition (E3) is immediate from [25, Lemma 5.9], which defines a total order β_1, \dots, β_n on the roots in $C_c(v)$ such that $E_c(\beta_i, \beta_j) = 0$ for $i < j$. We will quote three lemmas from [25] to establish conditions (E1) and (E2). They are parts of [25, Lemmas 3.1–3.3].

Lemma 5.20. *If s is initial in c and β_t is any positive root, then $E_c(\alpha_s, \beta_t) \geq 0$, with equality if and only if $t \in W_{\langle s \rangle}$.*

Lemma 5.21. *If s be final in c and β_t is any positive root, then $E_c(\beta_t, \alpha_s) \geq 0$, with equality if and only if $t \in W_{\langle s \rangle}$.*

Lemma 5.22. *If s is initial or final in c , then $E_c(\beta, \beta') = E_{scs}(s\beta, s\beta')$ for all β and β' in V .*

Proposition 5.23. *Condition (E1) holds for (Camb_c, C_c) .*

Proof. Our proof is by induction on the rank of W and the length of v . The statement is vacuously true when $\text{Rank}(W) = \ell(v) = 0$, as $C_c(v)$ is empty.

Let β and γ be in $C_c(v)$ with $\text{sgn}(\beta) = 1$ and $\text{sgn}(\gamma) = -1$. Let s be initial in c . **Case 1:** $v \not\geq s$. In this case $v \in W_{\langle s \rangle}$ and $C_c(v) = C_{sc}(v) \cup \{\alpha_s\}$, so γ is in $C_{sc}(v)$ and thus equals $-\beta_t$ for some $t \in W_{\langle s \rangle}$. If β is α_s , then Lemma 5.20 says that $E_c(\beta, \gamma) = 0$. Otherwise, β is in $C_{sc}(v)$, and $E_{sc}(\beta, \gamma) = 0$ by induction on rank. It is immediate from the definition that E_{sc} is the restriction of E_c , so $E_c(\beta, \gamma) = 0$.

Case 2: $v \geq s$. The roots $\pm\alpha_s$ switch signs when acted on by s , but no other roots change sign when acted on by s . It is impossible to have $\beta = \alpha_s$, because if so, $\text{Cone}_c(v)$ and vD are on opposite sides of the hyperplane α_s^\perp in V^* . This contradicts Theorem 5.8. Thus $s\beta$ is a positive root.

If $\gamma \neq -\alpha_s$, then $s\gamma$ is a negative root. The roots $s\beta$ and $s\gamma$ are in $C_{scs}(sv)$, so by induction on $\ell(v)$, we see that $E_{scs}(s\beta, s\gamma) = 0$. Lemma 5.22 now says that $E_c(\beta, \gamma) = 0$. If $\gamma = -\alpha_s$ then s is a cover reflection of v by Proposition 5.4. Proposition 5.19 says that $\beta = s\beta'$ for some positive root $\beta' \in C_{sc}(v_{\langle s \rangle})$. Lemma 5.22 says that $E_c(\beta, \gamma) = E_{scs}(s\beta, s\gamma)$, which can be rewritten as $E_{scs}(\beta', \alpha_s)$, which is zero by Lemma 5.21. \square

Proposition 5.24. *Condition (E2) holds for (Camb_c, C_c) .*

Proof. Our statement is by the usual induction on length and rank; it is vacuously true for rank < 2 .

Let β and γ be distinct roots in $C_c(v)$ with $\text{sgn}(\beta) = \text{sgn}(\gamma)$. Since Condition (E3) holds, we know that either $E_c(\beta, \gamma) = 0$ or $E_c(\gamma, \beta) = 0$. Thus, in light of Proposition 2.10, when it is convenient, we can verify that $K(\beta, \gamma) \leq 0$ to show that $E_c(\beta, \gamma) \leq 0$.

Case 1: β and γ are both negative roots. This case can be handled without induction. Proposition 5.4 says that β and γ are both associated to cover reflections of v . Thus $\beta = v\alpha_p$ and $\gamma = v\alpha_q$, for p and $q \in S$. But K is invariant under the action of W , so $K(\beta, \gamma) = K(\alpha_p, \alpha_q) \leq 0$.

Case 2: β and γ are both positive roots.

Let s be initial in c .

Case 2a: $v \not\geq s$. In this case $v \in W_{\langle s \rangle}$ and $C_c(v) = C_{sc}(v) \cup \{\alpha_s\}$. If neither β nor γ equals α_s , then both are in $C_{sc}(v)$, and $E_c(\beta, \gamma) = E_{sc}(\beta, \gamma)$, which is nonpositive by induction on rank. If $\beta = \alpha_s$ then γ is in $C_{sc}(v)$ and so Lemma 5.20 says that $E_c(\beta, \gamma) = 0$. If $\gamma = \alpha_s$, then β is a positive root in $C_{sc}(v)$. Thus β is a positive combination of simple roots α_r with $r \neq s$, and $K(\alpha_r, \alpha_s) \leq 0$ for each such r , so $K(\beta, \alpha_s) \leq 0$.

Case 2b: $v \geq s$. It is impossible to have $\beta = \alpha_s$ or $\gamma = \alpha_s$, because if so, we reach a contradiction to Theorem 5.8 as in Case 2 of the proof of Proposition 5.23. Thus $s\beta$ and $s\gamma$ are also positive. By induction on $\ell(v)$, we see that $E_{scs}(sv)(s\beta, s\gamma) \leq 0$, so Lemma 5.22 says that $E_c(\beta, \gamma) \leq 0$. \square

Remark 5.25. If W is finite, then we can approach Case 2 in a manner analogous to Case 1. Let $u = \pi_c^\uparrow(v)$, as defined in [23]. Then β and γ are of the form $u\alpha_p$ and $u\alpha_q$ for $p, q \in S$. But π_c^\uparrow cannot be defined for infinite Coxeter groups.

We have shown that (Camb_c, C_c) is a reflection framework. The fact that this framework is descending is a consequence of Proposition 5.4, as we now explain. The Positive labels condition follows because the identity element is the unique minimal element of the weak order, and thus every non-identity element has at least one cover reflection. The Half-edge condition follows because, if $\text{sgn}(C(v, e)) = -1$ then $C(v, e)$ is the root associated to a cover $v' \lessdot v$, and by construction, (v', v) is the edge e . The Descending chain condition follows for the same reason: every arrow $v \rightarrow v'$ corresponds to a cover $v' \lessdot v$. In particular, $\ell(v') < \ell(v)$. We have now completed the proof of Theorem 5.10.

5.3. Denominator vectors. We now comment on the problem of determining denominator vectors within a Cambrian framework. As of now, we only have a direct way of determining denominators in the case of finite Cartan type, where we rely on results of [22] and [24]. However, we conjecture that the same method works in arbitrary Cambrian frameworks.

In [22, Section 8], a map cl_c was defined, taking a c -sortable element to an n -tuple of roots. Here we give the same definition, modifying the notation slightly to allow us to reference individual roots in the n -tuple. Suppose $v \in W$ is c -sortable and let $a_1 \cdots a_k$ be its c -sorting word. Let $r \in S$. If r does not occur as a letter in $a_1 \cdots a_k$, then define $\text{cl}_c^r(v) = -\alpha_r$. If r occurs in $a_1 \cdots a_k$, then let i be the largest index such that $a_i = r$. The **last reflection** for r in v is $a_1 \cdots a_i \cdots a_1$. In this case, we define $\text{cl}_c^r(v)$ to be the positive root associated to the last reflection for r , that is to say, $\text{cl}_c^r(v) = a_1 \cdots a_{i-1} \alpha_r$. Write $\text{cl}_c(v)$ for $\{\text{cl}_c^r(v) : r \in S\}$.

As a consequence of [11, Theorem 1.9], when B is of finite Cartan type, the denominator vectors of cluster variables are all distinct. Thus in particular, the seeds in the exchange graph can be specified by the n -tuple of denominator vectors of the cluster variables in the seed. This n -tuple of denominator vectors, realized as roots as in Section 3.1, form a **combinatorial cluster**. The roots in the combinatorial cluster are all almost positive (see Section 2.2). The map from cluster variables to almost positive roots is a bijection. When W is finite, the map cl_c is a bijection from c -sortable elements to combinatorial clusters [22, Theorem 8.1]. The Cambrian fan \mathcal{F}_c , in the finite case, is a complete fan. On the other hand, the nonnegative linear span of each combinatorial cluster is a distinct n -dimensional simplicial cone, and these cones are the maximal cones of a complete simplicial fan [9, Theorem 1.10]. The map taking $\text{Cone}_c(v)$ to the nonnegative span of $\text{cl}_c(v)$ is a combinatorial isomorphism of fans [24, Theorem 1.1]. We now prove a more precise statement about denominator vectors.

Theorem 5.26. *Suppose B is of finite Cartan type. Let (Camb_c, C_c) be the Cambrian framework. If (v, e) is an incident pair in the graph Camb_c and x_e^v is the cluster variable assigned to (v, e) , then $\mathbf{d}(x_e^v)$ is the root $\text{cl}_c^r(v)$, where r is the element of S such that $C(v, e) = C_c^r(v)$.*

We now prepare to prove Theorem 5.26. First, we will need a lemma, which is immediate from the definitions, and which is a slightly more detailed version of [22, Lemma 8.5]. The lemma refers to a map σ_s , for $s \in S$. This is an involution on

almost positive roots defined by

$$\sigma_s(\alpha) = \begin{cases} \alpha & \text{if } \alpha \in (-\Pi) \text{ and } \alpha \neq -\alpha_s, \text{ or} \\ s(\alpha) & \text{otherwise.} \end{cases}$$

Lemma 5.27. *Let s be initial in c , let v be c -sortable and let $r \in S$. If $v \not\geq s$ then*

$$\text{cl}_c^r(v) = \begin{cases} -\alpha_s & \text{if } r = s, \text{ or} \\ \text{cl}_{sc}^r(v) & \text{if } r \neq s \end{cases}$$

If $v \geq s$ then $\text{cl}_c^r(v) = \sigma_s(\text{cl}_{scs}^r(sv))$.

Write ν_c for the linear map ν defined in Section 3.3. That is, for a simple root α_r , let

$$\nu_c(\alpha_r) = - \sum_{s \in S} E_c(\alpha_s^\vee, \alpha_r) \rho_s.$$

Let $R_c^r(v)$ denote the element dual to $(C_c^r)^\vee(v)$ in the dual basis to $C_c^\vee(v)$.

Proposition 5.28. *Let v be a c -sortable element and let $r \in S$. If $v \notin W_{\langle r \rangle}$ then $R_c^r(v) = \nu_c(\text{cl}_c^r(v))$.*

We prove this result for arbitrary acyclic B , not only in the special case where B is of finite Cartan type.

Proof. Let s be initial in c . If $v \not\geq s$ then $v \in W_{\langle s \rangle}$. Since $v \notin W_{\langle r \rangle}$, we conclude that $r \neq s$. By induction on the rank of W , the vector $R_{sc}^r(v)$ in $V_{\langle s \rangle}$ equals $\nu_{sc}(\text{cl}_{sc}^r(v))$, which equals $\nu_{sc}(\text{cl}_c^r(v))$ by Lemma 5.27. The coefficient of ρ_s in $\nu_c(\text{cl}_c^r(v))$ is zero by Lemma 5.20, since $\text{cl}_c^r(v)$ is a root associated to a reflection in $W_{\langle s \rangle}$.

Recall that $V_{\langle s \rangle}$ is the subspace of V spanned by the simple roots indexed by $\langle s \rangle = S \setminus \{s\}$. We identify the dual space $V_{\langle s \rangle}^*$ naturally with the subspace of V^* spanned by the fundamental weights indexed by $\langle s \rangle$. This subspace is the hyperplane α_s^\perp . Under this identification, the natural pairing $V_{\langle s \rangle}^* \times V_{\langle s \rangle} \rightarrow \mathbb{R}$ is the restriction of the natural pairing on V^* and V . Thus, $R_c^r(v)$ equals $\nu_{sc}(\text{cl}_c^r(v)) = \nu_c(\text{cl}_c^r(v))$, and we are done in this case.

If $v \geq s$ then, by the recursive definition of C_c^\vee , the vector $R_c^r(v)$ is $s \cdot R_{scs}^r(sv)$, where s acts on V^* by the action dual to its action on V . This action preserves ρ_r for $r \neq s$ and sends ρ_s to $-\rho_s - \sum_{q \in \langle s \rangle} K(\alpha_q^\vee, \alpha_s) \rho_q$.

We must treat separately the case where $sv \in W_{\langle r \rangle}$. Since $v \notin W_{\langle r \rangle}$, we have $r = s$ in this case. Furthermore, $\text{cl}_c^s(v) = \alpha_s$, so $\nu_c(\text{cl}_c^s(v))$ is $-\sum_{q \in S} E_c(\alpha_q^\vee, \alpha_s) \rho_q$. But applying Lemmas 2.10 and 5.20, $\nu_c(\text{cl}_c^s(v))$ is $-\rho_s - \sum_{q \in \langle s \rangle} K(\alpha_q^\vee, \alpha_s) \rho_q$. On the other hand, $R_{scs}^s(sv) = \rho_s$, so

$$R_c^s(v) = s\rho_s = -\rho_s - \sum_{q \in \langle s \rangle} K(\alpha_q^\vee, \alpha_s) \rho_q.$$

Finally, if $sv \notin W_{\langle r \rangle}$, then in particular s is not the last reflection for r in v , so $\text{cl}_{scs}^r(sv)$ is a positive root and $\text{cl}_c^r(v) = s \cdot \text{cl}_{scs}^r(sv)$ by Lemma 5.27. By induction on $\ell(v)$, the vector $R_{scs}^r(sv)$ is $\nu_{scs}(\text{cl}_{scs}^r(sv))$, which, by Lemmas 5.22 and 5.27 equals

$$\nu_{scs}(s \cdot \text{cl}_c^r(v)) = - \sum_{q \in S} E_{scs}(\alpha_q^\vee, s \cdot \text{cl}_c^r(v)) \rho_q = - \sum_{q \in S} E_c(s\alpha_q^\vee, \text{cl}_c^r(v)) \rho_q.$$

This can be rewritten as $-\sum_{q \in S} E_c(\alpha_q^\vee - K(\alpha_q^\vee, \alpha_s) \alpha_s^\vee, \text{cl}_c^r(v)) \rho_q$ and then as

$$\begin{aligned} &= E_c(\alpha_s^\vee, \text{cl}_c^r(v)) \rho_s - \sum_{q \in \langle s \rangle} E_c(\alpha_q^\vee - K(\alpha_q^\vee, \alpha_s) \alpha_s^\vee, \text{cl}_c^r(v)) \rho_q \\ &= -E_c(\alpha_s^\vee, \text{cl}_c^r(v)) \left(-\rho_s - \sum_{q \in \langle s \rangle} K(\alpha_q^\vee, \alpha_s) \rho_q \right) - \sum_{q \in \langle s \rangle} E_c(\alpha_q^\vee, \text{cl}_c^r(v)) \rho_q. \end{aligned}$$

This is $-E_c(\alpha_s^\vee, \text{cl}_c^r(v))(s \cdot \rho_s) - \sum_{q \in \langle s \rangle} E_c(\alpha_q^\vee, \text{cl}_c^r(v))(s \cdot \rho_q)$, which simplifies to $s \cdot \left(-\sum_{q \in S} E_c(\alpha_q^\vee, \alpha_s) \rho_q \right) = s \cdot \nu_c(\text{cl}_c^r(v))$. Thus $R_c^r(v) = \nu_c(\text{cl}_c^r(v))$ in this case as well. \square

We can now prove the theorem.

Proof of Theorem 5.26. The fact that the map taking $\text{Cone}_c(v)$ to the nonnegative span of $\text{cl}_c(v)$ is a combinatorial isomorphism of fans can be rephrased as follows: There exists a bijection from rays in the c -Cambrian fan to almost positive roots such that a set of rays spans a cone in the c -Cambrian fan if and only if the corresponding set of almost positive roots spans a cone in the fan of combinatorial clusters. (Indeed, this rephrasing is closer to the way that [24, Theorem 1.1] was proved.) The results of this paper give another such bijection: each ray in the Cambrian fan contains a unique vector $R(v, e)$ that is the \mathbf{g} -vector of x_e^v . This ray is mapped to the denominator vector of x_e^v .

We first verify that the two bijections are the same. The identity is a c -sortable element, and the rays of its c -Cambrian cone are spanned by the vectors $\rho_i : i \in I$, which are the \mathbf{g} -vectors of $x_i : i \in I$. The map cl_c , applied to the identity element, returns the set of negative simple roots, which are the denominator vectors of $x_i : i \in I$. For each $i \in I$, the element s_i is a c -sortable element adjacent to the identity in Camb_c . The cluster variable x_i is the one removed in the mutation from the initial seed to $\text{Seed}(s_i)$, and the negative simple root $-\alpha_i$ is the root removed from cl_c when moving from the identity element to s_i . Thus both bijections map the ray spanned by ρ_i to the negative simple root $-\alpha_i$. It is now easily to see that the two bijections coincide. (For example, one can show by induction on $\ell(v)$ that the rays of $\text{Cone}_c(v)$ are treated the same by both bijections, using the fact that each new v adds at most one new ray.)

It now remains to check the statement about the element r . Suppose v and v' are connected by an edge in Camb_c and suppose that $\text{cl}_c^r(v)$ is the denominator vector of the cluster variable that is removed in passing from the combinatorial cluster $\text{cl}_c(v)$ to the combinatorial cluster $\text{cl}_c(v')$. Proposition 5.28 implies that every other element β of $\text{cl}_c(v)$ is mapped by ν_c into $C_c^r(v)$ and thus that $\beta \in \text{cl}_c(v')$. We conclude that $\text{cl}_c^r(v)$ is the denominator vector of the cluster variable that is removed in passing from $\text{Seed}(v)$ to $\text{Seed}(v')$. This cluster variable is x_e^v . \square

The proof of Theorem 5.26 via Proposition 5.28 is the motivation for Conjecture 3.19. The proof also establishes the following result:

Theorem 5.29. *Conjecture 3.19 holds for B of finite Cartan type.*

Several conjectures follow from Conjecture 3.19 and Conjectures 3.11, 3.14, and 3.15, as explained in Section 3.3.

Corollary 5.30. *Conjectures 3.20, 3.21 and 3.22 hold for B of finite Cartan type.*

As a consequence of Theorem 5.29, the combinatorial clusters model contains complete information about \mathbf{g} -vectors, and therefore, in light of Theorem 3.24 and Corollary 3.31, about exchange matrices and principal coefficients as well. (Earlier results of Yang and Zelevinsky [27] already show that the combinatorial clusters model contains complete information about \mathbf{g} -vectors. These results combined with the result of [19] stated here as Corollary 3.28 also show that the combinatorial clusters model contains complete information about principal coefficients.)

Conjecture 3.19 implies the following conjecture about Cambrian frameworks.

Conjecture 5.31. *Theorem 5.26 holds even when B is not of finite Cartan type.*

Indeed, Conjecture 5.31 follows from Conjecture 3.19 because Proposition 5.28 identifies $\text{cl}_c^r(v)$ as $\eta_c(R_c^r(v))$ and Theorem 3.24(3) identifies $R_c^r(v)$ as the \mathbf{g} -vector of the appropriate cluster variable.

With Conjecture 5.31 unproven, we have no direct way, outside of finite Cartan type, of reading off denominator vectors from c -sortable elements. However, Proposition 5.28 is still useful in general, in that it provides a way to read off \mathbf{g} -vectors from c -sortable elements without computing a dual basis. Combining Proposition 5.28 with Theorem 3.24(3), we obtain the following theorem.

Theorem 5.32. *Let (Camb_c, C_c) be a Cambrian reflection framework. If (v, e) is an incident pair in the graph Camb_c and x_e^v is the cluster variable assigned to (v, e) , then $\mathbf{g}(x_e^v)$ is the weight $\nu_c \text{cl}_c^r(v)$, where r is the element of S such that $C(v, e) = C_c^r(v)$.*

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